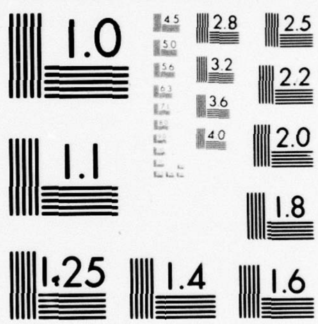
AD-A074 092 CALSPAN ADVANCED TECHNOLOGY CENTER BUFFALO NY F/G 1/3
DESIGN CRITERIA FOR OPTIMAL FLIGHT CONTROL SYSTEMS.(U)
SEP 79 K S GOVINDARAJ, E G RYNASKI, A T FAM N00014-78-C-0155
UNCLASSIFIED CALSPAN-6248-F-1 ONR-CR215-259-1F NL

1 OF 1
ADA 074092

END
DATE FILMED
10-79
DDC



MICROCOPY RESOLUTION TEST CHART

REPORT ONR-CR215-259-1F

LEVEL II

12

AD A074092

DESIGN CRITERIA FOR OPTIMAL FLIGHT CONTROL SYSTEMS

K. S. Govindaraj and E. G. Rynaski
Calspan Advanced Technology Center
Buffalo, New York 14225

and

A. T. Fam
State University of New York at Buffalo
Buffalo, New York 14214

Contract N00014-78-C-0155

ONR Task 215-259

7 September 1979

Final Report For Period Jan 78 - Nov 78

Approved for public release; distribution unlimited.

PREPARED FOR THE

OFFICE OF NAVAL RESEARCH • 800 N. QUINCY ST. • ARLINGTON • VA • 22217

SPECIAL NOTICES

Change of Address

Organizations receiving reports on the initial distribution list should confirm correct address. This list is located at the end of the report. Any change of address or distribution should be conveyed to the Office of Naval Research, Code 211, Arlington, VA 22217.

Disposition

When this report is no longer needed, it may be transmitted to other organizations. Do not return it the originator or the monitoring office.

Disclaimer

The findings and conclusions contained in this report are not to be construed as an official Department of Defense or Military Department position unless so designated by other official documents.

Reproduction

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 **REPORT DOCUMENTATION PAGE**

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 18 ONR-CR215-259-1F	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 DESIGN CRITERIA FOR OPTIMAL FLIGHT CONTROL SYSTEMS.		9 5. TYPE OF REPORT & PERIOD COVERED FINAL REPORT January 1978-November 1978 6. PERFORMING ORG. REPORT NUMBER 14 CALSPAN-6248-F-1
7. AUTHOR(s) 10 K. S. Govindaraj, E. G. Rynaski and A. T. Fam (SUNYAB)		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-78-C-0155
9. PERFORMING ORGANIZATION NAME AND ADDRESS Calspan Advanced Technology Center P.O. Box 400 Buffalo, New York 14225		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Project 6248-F
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 North Quincy Street Arlington, VA 22217 Contract to per contract loop		11 12. REPORT DATE 7 September 1979 13. NUMBER OF PAGES 90
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 95p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimal Control
Performance Index Weighting Matrices
Poles and Zeros
Multicontroller Design
Flying Qualities
Sequential Design Procedures

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~This report presents the results of~~ Discussed is the application of linear optimal control to the design of a multicontroller feedback system to satisfy aircraft flying qualities criteria. The problem addressed is that of relating the performance index weighting matrices to the poles and zeros of the closed-loop transfer functions. Two sequential design procedures, one computing the Riccati solution from a set of linear equations and the other computing the closed-loop eigenvectors, are presented that determine,

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

410 803 1B

20. ABSTRACT (cont.)

at each step, the pole-zero movements of the closed-loop transfer functions as the weighting matrix on the states is varied for a given weighting matrix on the controls. The performance index matrix constructed at each step to move the poles and zeros is added to get a final performance index matrix that moves the open-loop poles and zeros to more desirable locations. A control system design example with the X-22A V/STOL aircraft as the model, and using the first sequential design procedures, is presented.

Two alternative design techniques are also presented. The first ~~alternative design~~ is based upon the Riccati equation solution and the control weighting matrix rather than on the weighting matrices on the states and control, and in the second design technique, the change in the pole-zero locations is determined under perturbations in the performance index matrices.

FOREWORD

The study whose results are reported herein was performed under Contract N00014-78-C-0155 for the Office of Naval Research, Department of the Navy, by the Flight Research Branch of Calspan Corporation, Buffalo, New York, and the Systems Engineering Department of the State University of New York at Buffalo (SUNYAB) under a subcontract to Calspan Corporation. Mr. E. G. Rynaski was the overall Project Coordinator, Dr. K. S. Govindaraj was the Calspan Project Engineer and Dr. A. T. Fam headed the SUNYAB effort and contributed Section 4 of the report. The technical monitor for the Office of Naval Research was Mr. Robert VonHusen.

The authors wish to thank many members of the Flight Research Department for their encouragement during the course of this work, in particular Mr. Clarence Mesiah for the computational assistance, Mr. Robert Radford for his contribution to Section 5 of this report and Mr. Nha C. Nguyen, a graduate student at SUNYAB, for his assistance in the control systems design.

Finally, the authors wish to express their deep appreciation to Mrs. Janet Cornell and Mrs. Diane Thurn for their assistance during publication.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Section 1
INTRODUCTION

Air vehicles of the future, whether they be CTOL or V/STOL aircraft, will almost surely be control configured vehicles that make extensive use of active control technology. Multiple means of producing primary forces and moments on the vehicle for enhanced maneuverability will likely be commonplace. The technology to be used to conceptually design a multicontroller feedback system to satisfy flying qualities and other control system design criteria is an emerging technology. Almost always, the purpose of a feedback flight control or stability augmentation system is to enhance the flying qualities of the aircraft. To be most effective and responsive to the needs of Naval Aviation, one major thrust of flight control system design criteria should be toward the most efficient way to satisfy flying qualities requirements as defined by MIL-F-8785B(ASG) or MIL-F-83300, the only military CTOL or V/STOL specifications.

The objective of the research described in this paper is to investigate multicontroller control system design techniques that will satisfy flying qualities requirements. Augmented systems which do not increase the order of the closed loop system are desirable because they tend to avoid problems associated with phase shift and apparent time delays, they are less complex and easier to implement. The resulting dynamics of such augmented systems can be readily interpreted in terms of the current specifications on conventional phugoid, short period, Dutch roll, roll and spiral vehicle dynamic modes and the zeros of the transfer functions of substantial significance to the flight dynamics situation as defined by MIL-F-8785B or MIL-F-83300.

What is needed then, is a multicontroller design method that uses only passive feedback gains. The control system design of the near future will wish to achieve objectives in addition to satisfying flying qualities, such as bending moment and maneuver drag minimization or gust alleviation. Only by using optimal control methods can the control system designer hope to

obtain a multicontroller design that not only achieves objectives such as minimum maneuver loads and acceptable flying qualities as defined by MIL-F-8785B, but also does not, in general, increase the dynamic order of the system response.

A brief survey of the evolution of flight control design practices over the past fifteen years shows the following trends:

1. Flight control systems require more sensors that measure dynamic motions of the aircraft, resulting in an increased number of feedback paths.

2. Filter or compensation networks as command augmentation are introduced to modify pilot command inputs to the servos that drive the aerodynamic surfaces.

3. Filters, compensation networks and washout networks are being added in abundance to the flight control system.

4. "Inner loops" in the form of feedback to surfaces not directly commanded by the pilot are being proposed. An example would be a direct lift flap or spoiler programmed for gust alleviation.

These developments taken individually may be justified for individual airframes and flight tasks but the tendency is to build upon previous designs by cascading resulting in increased complexity and a very high order system.

The purpose of the research described in this report is to evolve flight control system design methods that are compatible with flying qualities specifications as they now exist and are potentially more systematic and powerful in the sense of achieving flight control objectives more quickly and more effectively. Linear optimal control is one of the most rapid and potentially most efficient methods of designing multi-input, multi-output systems.

Because the linear optimal control law inherently involves feedback through non-energy storage gains only, the order of the closed-loop response is not increased as compared to the open-loop response. Therefore, MIL-F-8785B can be used directly to determine the effect of the optimal control system on the resulting flying qualities of the aircraft. If flying qualities objectives can be directly included in the performance index to be traded off or evaluated along with active control objectives, then linear optimal control methods may be developed into an attractive and powerful tool for the flight control system designer.

Flying qualities requirements as presently specified by MIL-F-8785B define, among other things, acceptable regions for poles and zeros of transfer functions of specific response variables with respect to pilot command inputs. If linear optimal control is to be a viable design tool that can lead directly to flight control configurations satisfying flying qualities, then direct relationships connecting the performance index parameters with the resulting closed-loop poles and transfer function zeros should be available to the designer. This report directly addresses the problem of performance index selection and the effect of this selection on the closed-loop poles and transfer function zeros, the handling qualities parameters.

Linear optimal control methods have disadvantages. The feedback control law obtained by linear optimal methods in general requires feedback from each state to each controller, resulting in a complexity that is never necessary and never justifiable. Satisfactory flying qualities are defined for a fairly broad range of closed-loop dynamics, so a simplified closed-loop configuration seems always possible. Because a fully optimal feedback system can be considered to be of a complexity comparable to some recent high order flight control systems, a suboptimal system should yield a configuration that is simpler, lower order and have significantly better flying qualities.

Organization of the Report

Section two of this report briefly outlines the linear optimal control problem stressing the role played by transfer function and other

zeros in the optimal control problem and the effect of transfer function zeros on the closed-loop dynamic behavior of the optimal system. Section three describes two methods that might be further developed to obtain relationships between performance index parameters and zeros of closed-loop transfer functions. Section 4 describes two potentially useful alternative design techniques. Finally, design examples are given in Section 5 to illustrate the principles developed in earlier sections of the report.

Section 2
CHARACTERISTICS OF LINEAR OPTIMAL SYSTEMS

Linear optimal control, as generally applied to aircraft, involves the minimization of a quadratic form of scalar performance index

$$2V = \min_{u} \int_{0}^{\infty} (y^T Qy + u^T Ru)dt \qquad (2\text{-}1)$$

subject to the constraint of the familiar linearized, small perturbation equations of aircraft motion

$$\dot{x} = Fx + Gu \qquad (2\text{-}2a)$$

$$y = Hx \qquad (2\text{-}2b)$$

where x is a vector whose components represent the state or motion variables of the aircraft, u is a vector that represents the control input variables such as aileron, rudder and elevator deflection and y is a vector whose components usually represent either the measured quantities of the vehicle motion or the vehicle motions that are to be minimized by the performance index selection.

Many methods for solving this problem are in existence, but probably the most common and straightforward method is based upon a straightforward calculas of variations approach (Reference 3). A Lagrangian function is defined as

$$L = \frac{1}{2} (x^T H^T QHx + u^T Ru) + \lambda^T (-\dot{x} + Fx + Gu) \qquad (2\text{-}3)$$

where λ is a vector of Lagrange multipliers often called the adjoint state vector or costate of the system. The resultant Euler-Lagrange equations that must be satisfied to guarantee a minimum of the performance index are given by

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \text{or} \quad H^T QHx + F^T \lambda + \dot{\lambda} = 0 \qquad (2\text{-}4a)$$

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = 0 \quad \text{or} \quad Ru + G^T\lambda = 0 \tag{2-4b}$$

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0 \quad \text{or} \quad -\dot{x} + Fx + Gu = 0 \tag{2-4c}$$

For a non-negative definite output weighting matrix Q and positive definite weighting matrix R in the performance index of Equation (2-1), the solution is guaranteed stable. These are necessary and sufficient conditions for stability, but relaxation of these conditions need not yield an unstable closed-loop system. From Equation (2-4b) the optimal control law is obtained as

$$u = -R^{-1} G^T \lambda \tag{2-5}$$

It has been proven that λ is a linear function of x, i.e. $\lambda = Px$. Substitution of $\lambda = Px$ into equations (2-4) and a little algebra yields the familiar Riccati equation

$$0 = PF + F^TP - PGR^{-1}G^TP + H^TQH \tag{2-6}$$

whose positive definite symmetric solution yields the optimal feedback control law

$$u = -R^{-1}G^TPx \tag{2-7}$$

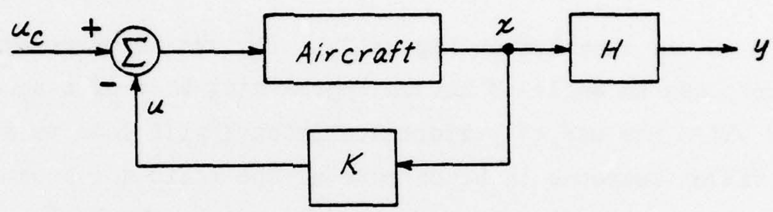
Effect of Performance Index Selection on Closed-Loop Dynamical Behavior

The optimal regulator as defined by the feedback control law of Equation (2-7) and the equations of motion (2-2) has an optimal response for any excitation to the system, whether the excitation be initial condition on the state, disturbance inputs or pilot command inputs. Without attempting to downplay or relegate the disturbance inputs to a lesser role, it is the command or pilot inputs that are most important to flying qualities.

Optimal control is a "black box", input-output design technique. When designing an optimal system to be commanded by a pilot, a more obvious method for definition of the equations of motion becomes

$$\dot{x} = Fx + G(u + u_c) \qquad (2\text{-}8)$$

which will yield a feedback configuration as indicated below:



The figure as indicated above suggests the classical servo problem of optimal control and one is tempted to define a performance index

$$u = \min_{u} \int_0^{\infty} \left[(x - u_c)^T Q(x - u_c) + u^T Ru\right] dt \qquad (2\text{-}9)$$

which would yield a gain matrix in the command part of the system shown in the above figure. However, the regulator or feedback part of the system is unaffected by the command input, so the two parts, regulator or feedback and command or feedforward parts for aircraft probably should be designed completely independently of each other. The feedforward part of the system will set static flying qualities requirements such as stick force per "g".

Minimization of an output or response variable has a very definite meaning in the optimal context. Every weighted output or combination of state variables in the performance index will have a closed loop response that approximates the response of a Butterworth filter as the weighting on the outputs becomes large relative to the weighting on the control variables.

For aircraft control, it then becomes important to recognize or define those motion variables or time histories that should tend to respond as a Butterworth filter responds, which is a smooth and well-behaved response

with little overshoot (ζ = 0.707 for a second order system). For an aircraft it is quite clear that this kind of response is not necessarily desirable for some aircraft motion variables. To illustrate, consider, as shown in Figure 2-1, the two performance indices, the resulting migration of the closed-loop poles of the two-degree-of-freedom aircraft representation as the weighting on the output variable becomes large with respect to the input, and the resulting responses of the aircraft to a pilot step command input.

As sketched in Figure 2-1, the use of performance index (1) will lead to a response in angle of attack approaching that of a second order Butterworth filter while the use of performance index (2) will lead to a first order Butterworth filter response in pitch rate as the ratio q/r becomes large. Therefore, a flight augmentation system designed on the basis of the first performance index will enable the pilot to change the angle of attack of his aircraft rapidly and smoothly by a step stick input, allowing for enhanced aircraft maneuverability. The second performance index will yield an aircraft that maneuvers sluggishly but allows for rapid and precise pitch rate and therefore attitude angle control.

The phenomena is easily explained on the basis of the differences between the locations of the zeros of the $\alpha/\delta_e(s)$ and the $\dot{\theta}/\delta_e(s)$ transfer functions. These transfer function zeros are fixed if feedback is provided for only one control surface, in this case the elevator. These zeros act, in the cases of Figure 2-1, as transmission zeros. For the performance index (2), the transmission zero is at L_α; for the performance index (1), the transmission zero is at ∞ (assuming zero L_{δ_e}). Therefore, a performance index containing positive value weighting measures of both $\alpha(t)$ and $\dot{\theta}(t)$ may have a transmission zero anywhere on the negative real axis to the left of L_α. But because the transfer function zeros cannot be altered, the response of the aircraft will be bounded by the two response sets shown in Figure 2-1 and fixed with respect to each other. If feedback were provided for multiple inputs as well as multiple outputs, the closed-loop transfer function zeros can be altered and closed-loop dynamic behavior can be altered much more freely. Therefore, before proceeding to the description of design methods to predict and

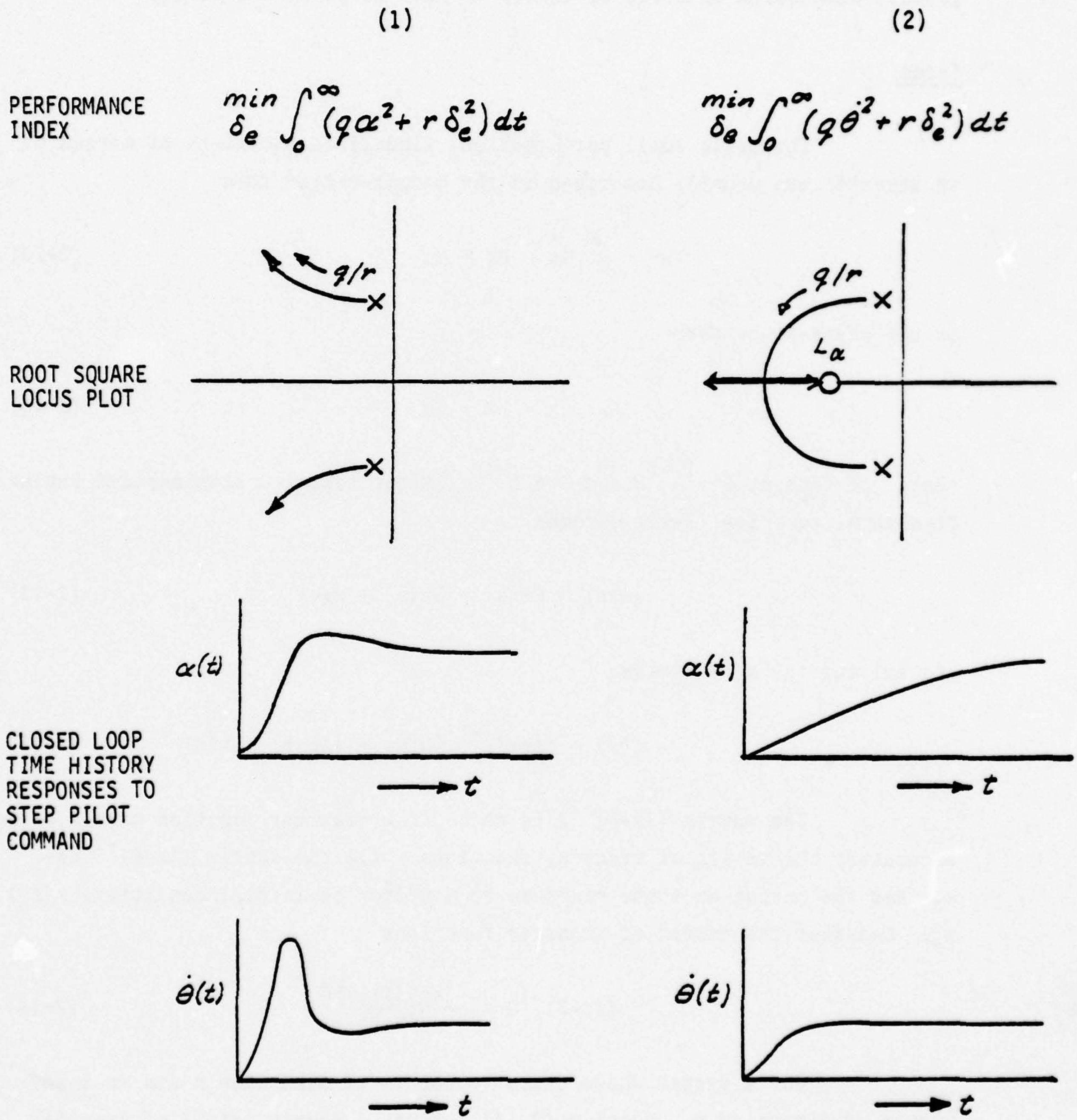


Fig. 2-1 RESPONSE TO STEP COMMAND

to place the zeros of closed-loop transfer functions for optimal systems, a general discussion of zeros of linear systems is presented first.

Zeros

The basic small perturbation, linearized equations of motion of an aircraft are usually described in the matrix-vector form

$$A\dot{x} = Bx + Cu \qquad (2\text{-}10)$$

or the state-space form

$$\dot{x} = Fx + Gu \qquad (2\text{-}11)$$

where, of course, $F = A^{-1}B$ and $G = A^{-1}C$. After taking a single-sided Laplace transform, equation (2-11) becomes

$$sx(s) = Fx(s) + Gu(s) + x(o) \qquad (2\text{-}12)$$

and solving for x(s) yields

$$x(s) = (Is-F)^{-1}\,Gu(s) + (Is-F)^{-1}\,x(o) \qquad (2\text{-}13)$$

The matrix $(Is-F)^{-1}G$ is called the transfer-function matrix,or more accurately the matrix of transfer functions while the matrix $(Is-F)^{-1}$ describes the output or state response to a vector of initial conditions $x(t_o) = x_o$. Consider the matrix of transfer functions

$$(Is-F)^{-1}G = \frac{\text{adj}\,[Is-F]\,G}{|Is-F|} \qquad (2\text{-}14)$$

For a system whose state vector is of dimension n and an input vector of dimension p, equation (2-14) defines a proper matrix of transfer functions, each transfer function has a numerator polynomial of order less than that of the denominator polynomial. The matrix adj $[Is-F]G$ defines the zeros

of the transfer function matrix. Each polynomial entry of adj $[IS-F]G$ defines zeros of individual transfer functions, while the determinant $[IS-F]$ defines the denominator polynomial common to all transfer functions and is called the characteristic polynomial. For a physical system such as an aircraft, the roots of the characteristic polynomial are called the poles of the system. By virtue of the fact that it can be flown, an aircraft may always be considered to be completely controllable and observable in the classical sense.

Transmission Zeros

The matrix of transfer functions defined by Equation (2-14) can be written as

$$(Is-F)^{-1}G = \frac{1}{|Is-F|}\begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1p} \\ N_{21} & & & \vdots \\ \vdots & & & \vdots \\ N_{n1} & \cdots & \cdots & N_{np} \end{bmatrix} = \frac{N(s)}{d(s)} \tag{2-15}$$

The determinant of each non-singular minor of the matrix $N(s)$ defines zeros of the transfer function matrix that can be different from the zeros of individual transfer functions or roots of each element of $N(s)$. For instance, a second minor of $N(s)$ is given by

$$\begin{vmatrix} N_{21}(s) & N_{23}(s) \\ N_{41}(s) & N_{43}(s) \end{vmatrix} = D(s)\, L^{2,4}_{1,3}(s) \tag{2-16}$$

where the notation $L^{2,4}_{1,3}$ is used to indicate that all rows except 2 and 4 have been deleted (upper index) and all columns except 1 and 3 (lower index) have been deleted. These zeros, originally called coupling zeros (References 4, 5) play an important role in the feedback design of multicontroller systems just as the zeros of transfer functions are important to single input control system designs.

The largest order non-singular minors of N(s), normally of dimension pxp, define the transmission zeros of the multivariable system. A pxp minor of N(s), called a p^{th} minor, is defined as

$$t(s) = \{D(s)\}^{p-1} \; L(s)^{\text{any p rows}}_{1,2,\ldots p} \qquad (2\text{-}17)$$

A combination of the zeros of the $(n-p)^{th}$ order polynomials $L(s)^{\text{any p rows}}_{1,2,\ldots p}$ defined by Equation (2-8) are called transmission zeros of the system and are invariant with feedback. These transmission zeros define, in general, the terminal or high feedback gain limiting locations of n-p closed-loop poles and the zeros, including both transfer function zeros (first minors of N(s)) and all other zeros as well (2^{nd} to $(p-1)^{th}$ minors of N(s)).

Therefore, a system that has as many outputs as inputs, i.e.

$$\dot{x} = Fx + Gu \qquad (2\text{-}18a)$$

$$y = Hx \qquad (2\text{-}18b)$$

where u is an input vector of dimension p and y is an output vector of dimension p, will have a square transfer function matrix N(s). The determinant of N(s), $|N(s)|$, is given by

$$|N(s)| = H\,[Is - F]^{-1}\,G$$
$$= \{D(s)\}^{p-1} \cdot L(s)^{1,\ldots.p}_{1,\ldots.p} \qquad (2\text{-}19)$$

and will be the only term defining the system transmission zeros. If G and H are square and nonsingular, then L(s) is a constant and there are no finite transmission zeros. In this case, poles and possibly some transfer function zeros can be placed anywhere with feedback.

The transmission zeros of a single controller system are then the zeros of the output of the system. If only one measurement of the system dynamics is used for feedback, then the transmission zeros are the transfer function zeros of that measurement. If two or more outputs are used for feedback, as

$$u_1 = k_1 y_1 + k_2 y_2$$

then the control law can be written as

$$u = k_1 y_1 + \left(\frac{k_2}{k_1}\right) y_2 = k_1 \hat{y}$$

and the transmission zeros are obtained from the numerator polynomial of transfer function of the equivalent output

$$\frac{\hat{y}}{u}(s) = \frac{y_1(s) + k_2/k_1 \, y_2(s)}{u(s)} \qquad (2\text{-}20)$$

Zeros, then, are as important to the design of a feedback control system as the poles of the system. As an example of the effect of transmission zeros on the closed loop poles and transfer function zeros of a multivariable system, consider the two-input, three output system schematically shown in Figure 2-2 below,

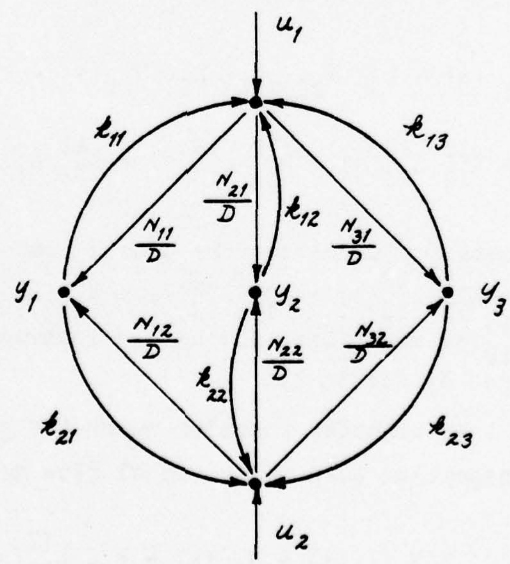


Fig. 2-2 GENERALIZED 2 INPUT, 3 OUTPUT SYSTEM

From the figure, it can be seen that the feedback control law is given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (2\text{-}21)$$

so each of the outputs is fed back into both inputs. Using the rules of signal flow theory, the closed loop characteristic polynomial can be expressed as

$$\begin{aligned} \Delta(s) = {} & D(s) - k_{11}\, N_{11}(s) - k_{22}\, N_{22}(s) - k_{12}\, N_{21}(s) - k_{21}\, N_{12}(s) - k_{13}\, N_{31}(s) \\ & -k_{23}\, N_{32}(s) + (k_{11}\, k_{22} - k_{12}\, k_{21})\, L_{12}^{12}(s) + (k_{11}\, k_{23} - k_{13}\, k_{21})\, L_{12}^{13}(s) \\ & + (k_{22}\, k_{13} - k_{12}\, k_{23})\, L_{12}^{23}(s) \end{aligned} \qquad (2\text{-}22)$$

or, using the notation K_{cd}^{ab} to indicate the minor of the feedback gain matrix that retains the a, b rows and c, d columns, Equation (2-22) can be written

$$\begin{aligned} \Delta(s) = {} & D(s) - k_{11}\, N_{11}(s) - k_{22}\, N_{22}(s) - k_{12}\, N_{21}(s) - k_{21}\, N_{12}(s) - k_{13}\, N_{31}(s) \\ & -k_{23}\, N_{32}(s) + K_{12}^{12}\, L_{12}^{12}(s) + K_{13}^{12}\, L_{12}^{13}(s) + K_{23}^{12}\, L_{12}^{23}(s) \end{aligned} \qquad (2\text{-}23)$$

The transmission zeros for the closed loop poles are defined from the expression $T(s) = K_{12}^{12}\, L_{12}^{12}(s) + K_{13}^{12}\, L_{12}^{13}(s) + K_{23}^{12}\, L_{12}^{23}(s)$ and the zeros defined by $L_{12}^{12}(s)$, $L_{12}^{13}(s)$ and $L_{12}^{23}(s)$ are called invariant zeros of the system which cannot be changed by feedback.

A closed loop transfer function numerator polynomial, say $N_{11_{c\ell}}(s)$, can be obtained by inspection also using signal flow methods

$$N_{11_{c\ell}}(s) = N_{11}(s) - k_{22}\, L_{12}^{12}(s) - k_{23}\, L_{12}^{13}(s) \qquad (2\text{-}24)$$

which shows how the transfer function zeros vary as a function of the second minors of the transfer function matrix and the feedback gains from other outputs to other inputs, in this case the feedback from outputs 2 and 3 to input 2.

Equation (2-24) can be expressed in the classical root locus form to obtain the closed-loop zeros as

$$0 = 1 - k_{22} \frac{L_{12}^{12}(s)}{N_{11}(s)} \left(1 + \frac{k_{23}}{k_{22}} \frac{L_{12}^{13}(s)}{L_{12}^{12}(s)} \right) \qquad (2\text{-}25)$$

which shows that two separate but consecutive root locus plots, one involving the parameter $\frac{k_{23}}{k_{22}}$ and the other involving k_{22} are required to design for a specific closed-loop transfer function set of zeros. Equation (2-25) shows that for a fixed ratio $\frac{k_{23}}{k_{22}}$ the transmission zeros for the numerator polynomial $N_{11_{c\ell}}(s)$ are obtained from the roots of the polynomial $k_{22} L_{12}^{12}(s) + k_{23} L_{12}^{13}(s) = 0$.

Unfortunately, the expressions for the definition of closed-loop zeros of transfer functions of linear systems designed using linear optimal control methods are not obtainable as easily as indicated above, but progress leading to these design aids has been made. This progress has been described in the following section.

Section 3
OPTIMAL CONTROL SYSTEMS DESIGN PROCEDURES

3.1 GENERAL PROBLEM

The linear optimal control problem as given in Section 2 is summarized here briefly.

Linear optimal control, as generally applied to aircraft, involves the minimization of a quadratic form of scalar performance index

$$J = \frac{1}{2}\int_0^\infty (y^T Q_y y + u^T R u)dt \tag{3-1}$$

Subject to the constraint of the familiar linearized, small perturbation equations of aircraft motion

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned} \tag{3-2}$$

The performance index can be written in terms of states x as

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt \tag{3-2a}$$

where $Q = H^T Q_y H$

The solution to this problem is given by the Euler-Lagrange equations

$$\begin{aligned} \dot{x} &= Fx + Gu \\ \dot{p} &= -F^T p - Qx \\ Ru + G^T p &= 0 \end{aligned} \tag{3-3}$$

The adjoint or the costate is a linear function of x, i.e. $p = Px$. The optimal feedback control law is given by

$$u = -R^{-1}G^T Px \tag{3-4}$$

and the matrix P is the positive semidefinite solution of the Riccati equation

$$PF + F^TP - PGR^{-1}G^TP + H^TQH = 0 \tag{3-5}$$

After taking the single-sided Laplace transform of Equation (3-3), the result is

$$\begin{bmatrix} sI-F & GR^{-1}G^T \\ -Q & -sI-F^T \end{bmatrix} \begin{bmatrix} x(s) \\ p(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix} \tag{3-6}$$

The characteristic equation of the closed-loop set of Equations (3-6) is given by the determinant expression

$$\Delta(-s)\ \Delta(s) = \begin{vmatrix} sI-F & GR^{-1}G^T \\ -Q & -sI-F^T \end{vmatrix} \tag{3-7}$$

The scalar expression that defines the locus of poles of the closed-loop system and its adjoint is given by the well known root square locus expression as given in Section 2.

$$\left| I + R^{-1}G^T \left[-sI - F^T\right]^{-1} Q \left[sI - F\right]^{-1} G \right| \tag{3-8}$$

The feedback gain matrix K required to obtain the closed-loop dynamics is given by

$$K = -R^{-1}G^TP$$

and the optimal closed-loop system becomes

$$\dot{x} = (F - GK)\ x + Gu_c \tag{3-9}$$

u_c is the command input.

The characteristic equation of the optimal system is given by

$$|sI - F + GK| = 0 \tag{3-10}$$

and the closed-loop transfer functions are given by

$$x(s) = \frac{\text{adj}\,[sI-F+GK]}{|sI - F + GK|} G\, u_c(s)$$

$$= W(s)\, u_c(s) \tag{3-11}$$

From Equation (3-6) the optimal state and costate are given by

$$\begin{bmatrix} X(s) \\ p(s) \end{bmatrix} = \begin{bmatrix} sI-F & GR^{-1}G^T \\ -Q & -sI-F^T \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix}$$

$$= \frac{\text{adj}\begin{bmatrix} sI-F & GR^{-1}G^T \\ -Q & -sI-F^T \end{bmatrix}}{\Delta(s)\ \Delta(-s)} \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix} \tag{3-12}$$

To get the closed-loop transfer functions from Equation (3-12), the state and the costate have to be separated either through spectral factorization or through the Riccati solution. Once the states and costates are separated, the problem then is to relate the closed-loop transfer functions to the performance index matrices. In this section, two sequential design procedures will be described which determine pole-zero movements of the transfer functions as the weighting matrix on the states is varied for a given weighting matrix on the control.

3.2 SEPARATION OF STATE AND COSTATE

The solution of the linear optimal control problem requires the solution of Euler-Lagrange equations (Hamiltonian Equations)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \mathcal{H} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{3-13}$$

Following the development given in Reference 7, Equation (3-13) can be transformed into the diagonal form by the transformation

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{3-14}$$

where the matrix

$$\underset{\sim}{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is a 2nx2n matrix of eigenvectors and can be found from

$$\left[sI - \mathcal{H}\right] T_i \Big|_{s = s_i} = 0 \tag{3-15}$$

where s_i is the i^{th} eigenvalue of the optimal system and its adjoint and T_i is the i^{th} eigenvector of the dynamic matrix $\mathcal{H}$.

The solution in the transformed domain is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{3-16}$$

The response can then be written

$$\begin{bmatrix} \mathfrak{z}_1(t) \\ \mathfrak{z}_2(t) \end{bmatrix} = \begin{bmatrix} e^{\Lambda t} & 0 \\ 0 & e^{-\Lambda t} \end{bmatrix} \begin{bmatrix} \mathfrak{z}_1(0) \\ \mathfrak{z}_2(0) \end{bmatrix} \tag{3-17}$$

or

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} e^{\Lambda t} & 0 \\ 0 & e^{-\Lambda t} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ p(0) \end{bmatrix} \tag{3-18}$$

Let the inverse of the transformation matrix be

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}^{-1} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \tag{3-19}$$

then

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} e^{\Lambda t} & 0 \\ 0 & e^{-\Lambda t} \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} x(0) \\ p(0) \end{bmatrix} \tag{3-20}$$

from which x(t) can be obtained as

$$x(t) = T_{11}\, e^{\Lambda t} \cdot Kx(0) + T_{11}\, e^{\Lambda t}\, L\, p(0) + T_{12}\, e^{-\Lambda t}\, M\, x(0) + T_{12}\, e^{-\Lambda t}\, N\, p(0) \tag{3-21}$$

The optimal solution for x(t), which is stable, cannot contain terms in $e^{-\Lambda t}$. Therefore, x(0) and p(0) must be related by the expression

$$P(0) = -N^{-1}\, M\, x(0) \tag{3-22}$$

The optimal solution now is given by

$$x(t) = T_{11} e^{\Lambda t} (K - LN^{-1}M)\, x(0) \tag{3-23}$$

From the identity

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} = \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3-24}$$

the following relationships can be obtained

$$M = -NT_{21} T_{11}^{-1} \qquad K = -LT_{22} T_{12}^{-1}$$

$$N = \left[T_{22} - T_{21} T_{11}^{-1} T_{12}\right]^{-1} \qquad L = \left[T_{21} - T_{22} T_{12}^{-1} T_{11}\right]^{-1} \tag{3-25}$$

Substituting these expressions for K, L, M, and N, the optimal solution is given by

$$x(t) = T_{11} e^{\Lambda t} T_{11}^{-1} x(0) \tag{3-26}$$

The closed-loop transfer functions are given by

$$x(s) = T_{11} \left[sI - \Lambda\right]^{-1} T_{11}^{-1} G\, u_c(s) \tag{3-27}$$

Therefore, once the eigenvectors of the Hamiltonian system of Equations (3-13) are determined, the feedback control law can be computed directly. The solution of the Riccati equation is given by

$$P = T_{21} T_{11}^{-1} \tag{3-28}$$

and the feedback control law is given by

$$u = -R^{-1} G^T T_{21} T_{11}^{-1} x \tag{3-29}$$

Equation (3-27) shows that the closed-loop transfer functions are directly related to the eigenvalues, eigenvectors and control matrix G of the system. These eigenvalues and eigenvectors are a function of the weighting matrices Q and R in the performance index. A functional relationship between Q, R and the eigenvector matrix would be very useful in the selection of performance indices for multicontroller linear optimal control systems.

The first n columns of the matrix of eigenvectors

$$T = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right]$$

are the eigenvectors corresponding to the left half plane eigenvalues and the last n columns are the eigenvectors corresponding to the right half plane eigenvalues. Let the 2nx1 vector

$$t = \left[\begin{array}{c} t_1 \\ \hline t_2 \end{array}\right] \tag{3-30}$$

where t_1 and t_2 are nx1 vectors, represent the closed-loop eigenvectors of the Hamiltonian matrix and let μ represent the closed-loop eigenvalue. It follows that

$$\left[\begin{array}{c|c} F & -GR^{-1} G^T \\ \hline -Q & -F^T \end{array}\right] \left[\begin{array}{c} t_1 \\ \hline t_2 \end{array}\right] = \mu \left[\begin{array}{c} t_1 \\ \hline t_2 \end{array}\right] \tag{3-31}$$

or

$$Ft_1 - GR^{-1} G^T t_2 = \mu t_1$$
$$-Qt_1 - F^T t_2 = \mu t_2 \qquad (3\text{-}32)$$

Solving for t_2 we get

$$t_2 = \left[-\mu I - F^T \right]^{-1} Qt_1 \qquad (3\text{-}33)$$

Substituting for t_2 we get

$$\left\{ F - GR^{-1} G^T \left[- \mu I - F^T \right]^{-1} Q \right\} t_1 = \mu t_1 \qquad (3\text{-}34)$$

If from Equation (3-34), the optimal closed-loop eigenvectors can be determined as a function of the weighting matrices of the performance index, then the closed-loop transfer functions can be determined as a function of the weighting matrices.

Another way of separating the states and costates is to find the matrix P which relates states and costates.

In Reference 1, Whitbeck describes a conceptual solution to the linear optimal control problem without recourse to the steady-state Riccati equation

$$PF + PF^T + Q - PGR^{-1} G^T P = 0$$

The problem is that of finding the set of numbers which define the adjoint or costate vector p at time zero. After taking the Laplace transform of the Hamiltonian equations, the solution in the Laplace domain is given by

$$\begin{bmatrix} x(s) \\ p(s) \end{bmatrix} = \begin{bmatrix} sI - F & GR^{-1} G^T \\ -Q & -sI - F^T \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix} \qquad (3\text{-}35)$$

or

$$\begin{bmatrix} x(s) \\ p(s) \end{bmatrix} = A(s) \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix} \qquad (3\text{-}36)$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{3-37}$$

and the nxn matrices A_i are obtained by inverting the 2nx2n matrix in Equation (3-35) in the partitioned form and are given by

$$A_1 = \left\{ I + \left[sI - F\right]^{-1} GR^{-1} G^T \left[- sI - F^T\right]^{-1} Q \right\}^{-1} \left[sI - F\right]^{-1}$$

$$A_2 = \left[sI - F\right]^{-1} GR^{-1} G^T \left\{ I + \left[- sI - F^T\right]^{-1} Q \left[sI - F\right]^{-1} GR^{-1} G^T \right\}^{-1} \left[- sI - F^T\right]^{-1}$$

$$A_3 = \left\{ I + \left[- sI - F^T\right]^{-1} Q \left[sI - F\right]^{-1} GR^{-1} G^T \right\}^{-1} \left[- sI - F^T\right]^{-1} Q \left[sI - F\right]^{-1}$$

$$A_4 = - \left\{ I + \left[-sI - F^T\right]^{-1} Q \left[sI - F\right]^{-1} GR^{-1} G^T \right\}^{-1} \left[-sI - F^T\right]^{-1} \tag{3-38}$$

The solution to the Hamiltonian equations may also be written as

$$\begin{bmatrix} x(s) \\ P(s) \end{bmatrix} = \frac{1}{\Delta(s)\ \Delta(-s)} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} x(0) \\ -p(0) \end{bmatrix} \tag{3-39}$$

where $B_i = \Delta(s)\ \Delta(-s)\ A_i$ (3-40)

and

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \text{adj} \left[sI - \mathcal{H}\right]$$

The elements of the matrix B are polynomials in s. Any component of the state vector, say, for example, $x_1(s)$ can be written as

$$x_1(s) = \frac{b_{11}\,x_1(0) + b_{12}\,x_2(0) + \ldots .\, b_{1n}\,x_n(0)}{\Delta(s)\;\Delta(-s)}$$

$$+ \quad \frac{b_{1n+1}\,p_1(0) + b_{1n+2}\,p_2(0) + \ldots .\, b_{12n}\,p_n(0)}{\Delta(s)\;\Delta(-s)} \tag{3-41}$$

where $b_{1j}(s)$, $j = 1, 2, \ldots .\, 2n$, represent the elements in the first row of the B matrix. If the optimal solution $x_1(s)$ is to represent a stable component of the closed-loop state vector, then $p_1(0), \ldots .p_n(0)$ must be such that $\Delta(-s)$ cancels identically into the numerator of Equation (3-40). The numerator of Equation (3-40) must equal zero for those values of s such that $\Delta(-s) = 0$. The values of s which make $\Delta(-s) = 0$ are the right half plane eigenvalues and n linear equations in n unknowns are obtained relating x(0) to p(0) when s takes on these values. By solving these linear equations p(0) may be expressed in terms of x(0) as

$$p(0) = Px(0)$$

where P must be identically equal to the positive semi-definite solution of the steady-state Riccati equation.

The n linear equations in n unknowns were obtained by evaluating the numerator of one component of the state vector at the right half eigenvalues. The same set of equations can be obtained by evaluating the numerator of each component of the state vector at a right half plane eigenvalue. The state vector is given by

$$x(s) = B_1\,x(0) - B_2\,p(0) \tag{3-42}$$

The n linear equations relating the x(0) and p(0) may be obtained by evaluating each row of the matrices B_1 and B_2 at a right half plane eigenvalue. The matrix P is given by

$$P = B_2^{-1} B_1 \Big|_{s=s_i} \quad i = 1, 2, \ldots n \tag{3-43}$$

where s_i are the right half plane eigenvalues.

The matrices B_1 and B_2 are functions of Q and R. The closed-loop transfer functions are given by

$$x(s) = \left[B_1 - B_2 P\right] G \; u_c(s) \tag{3-44}$$

An example is given to illustrate the concept involved. In the example considered in Reference 1

$$F = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R^{-1} = \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \tag{3-45}$$

The Hamiltonian matrix, after taking the Laplace transform, becomes

$$\left[sI - \mathcal{H}\right] = \begin{bmatrix} s & -1 & 7 & 0 \\ 2 & s+3 & 0 & 4 \\ 0 & 0 & -s & 2 \\ 0 & -5 & -1 & -s+3 \end{bmatrix} \tag{3-46}$$

By inverting the matrix $\left[sI - \mathcal{H}\right]$, the first component of the state vector $x_1(s)$ is obtained and the set of linear equations relating x(0)

and p(0) can be obtained by evaluating the numerators of $x_1(s)$ at the closed-loop right half plane eigenvalues, $s = 3$ and $s = 4$ yielding the following set of equations

$$\begin{bmatrix} 3 & -2 \\ \frac{5}{2} & -3 \end{bmatrix} \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{3-47}$$

The same set of linear equations (3-47) can be obtained by substituting $s = 3$ in the numerator of $x_1(s)$ and $s = 4$ in the numerator of $x_2(s)$. The adjoint of $[sI - \mathcal{H}]$ is given by

$$\text{adj}\,[sI - \mathcal{H}] = \begin{bmatrix} s^3 - 27s + 6 & s^2 - 3s + 72 & 7s^2 - 207 & 18s + 42 \\ -2s^2 + 6s - 4 & s^3 - 3s^2 + 2s & -18s + 42 & 4s^2 - 28 \\ 10 & -10s & -s^3 + 27s + 6s & -4s^2 - 6s - 4 \\ 10s & -5s^2 & s^2 + 3s + 72 & -s^3 - 3s^2 - 2s \end{bmatrix} \tag{3-48}$$

Substitution of s = 3 in the first row of the matrix adj $[sI - \mathcal{H}]$ and s = 4 in the second row yields the same set of equations (3-47) and

$$\begin{bmatrix} p_1(0) \\ P_2(0) \end{bmatrix} = \begin{bmatrix} .25 & -.125 \\ -.125 & .5625 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = P\,x(0) \tag{3-49}$$

3.3 SEQUENTIAL DESIGN PROCEDURES

A design procedure has been developed by Solheim (Reference 2) that finds the elements of the weighting matrix of a performance index that correspond to the given poles. With a given control weighting matrix R, the procedure enables the determination of a Q matrix so that the closed-loop system has prescribed eigenvalues. The procedure is sequential so that only one pole is moved at a time. A performance index matrix Q is determined so that only one pole changes. Each time a pole is moved, a performance index matrix Q

is determined that corresponds to the change in the pole location. Finally, all the different weighting matrices are added together to yield one weighting matrix that corresponds to prescribed closed-loop pole locations. The procedure for pole placement with distinct real poles is described briefly below.

3.3.1 Pole Placement

The system of equations (3-1) is transformed into the Jordan form with the transformation $x = T\mathfrak{z}$ yielding the equation

$$\dot{\mathfrak{z}} = \Lambda \mathfrak{z} + T^{-1} Gu \tag{3-50}$$

where Λ is assumed to be a diagonal matrix because the multiple eigenvalue (3-2a) problem does not normally exist for a finite time in the aircraft control problem.

The performance index (3-2a) can be expressed in terms of the transformed state vector $\mathfrak{z}$:

$$J = \frac{1}{2} \int_0^\infty (\mathfrak{z}^T T^T QT\mathfrak{z} + u^T Ru)\, dt$$

$$= \frac{1}{2} \int_0^\infty (\mathfrak{z}^T \tilde{Q} \mathfrak{z} + u^T Ru)\, dt \tag{3-51}$$

where $\tilde{Q} = T^T QT$

The value of the performance index J is same as before. Selection of a $\tilde{Q}$ matrix will determine a Q matrix through the transformation matrix T keeping the value of J the same under transformation.

The Hamiltonian set of equations are now given by

$$\begin{bmatrix} \dot{\mathfrak{z}} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} \Lambda & -C \\ -\tilde{Q} & -\Lambda \end{bmatrix} \begin{bmatrix} \mathfrak{z} \\ \tilde{p} \end{bmatrix} = \check{\mathcal{H}} \begin{bmatrix} \mathfrak{z} \\ \tilde{p} \end{bmatrix} \tag{3-52}$$

where $C = T^{-1} GR^{-1} G^T T^{-T}$ and T^{-T} is the inverse of the transpose.

The linear optimal control law is given by

$$u = -R^{-1} G^T T^{-T} \tilde{P} \tag{3-53}$$

where $\tilde{P}$ is the appropriate Riccati solution.

The eigenvalues of $\mathcal{H}$ and $\check{\mathcal{H}}$ are identical and are given by the characteristic equation

$$|sI - \check{\mathcal{H}}| = 0 \qquad (3\text{-}54)$$

using the expression as given (Reference 2):

$$\begin{bmatrix} I & 0 \\ -\tilde{Q}\,[sI - \Lambda]^{-1} & I \end{bmatrix} [sI - \check{\mathcal{H}}] = \begin{bmatrix} sI - \Lambda & C \\ 0 & sI + \Lambda - \tilde{Q}\,[sI - \Lambda]^{-1} C \end{bmatrix} \qquad (3\text{-}54a)$$

The characteristic equation is obtained as

$$|sI - \check{\mathcal{H}}| = |sI - \Lambda|\,|sI + \Lambda - \tilde{Q}\,[sI - \Lambda]^{-1} C| \qquad (3\text{-}55)$$

If only one element of the matrix $\tilde{Q}$ is non-zero, namely $\tilde{q}_{jj}$, the second determinant on the right-hand side of (3-55) becomes

$$|sI + \Lambda - \tilde{Q}\,[sI - \Lambda]^{-1} C| = \begin{vmatrix} s+\lambda_1 & 0 & - - - - & - - - - & 0 \\ 0 & s+\lambda_2 & - - - - & - - - - & 0 \\ | & & & & | \\ \dfrac{-\tilde{q}_{jj}C_{j1}}{s-\lambda_j} & - - - - & & s+\lambda_j - \dfrac{\tilde{q}_{jj}C_{jj}}{s-\lambda_j} & \dfrac{-\tilde{q}_{jj}C_{jn}}{s-\lambda_j} \\ | & & & & | \\ 0 & & & & s+\lambda_n \end{vmatrix} \qquad (3\text{-}56)$$

The characteristic equation may then be written as

$$|sI - \check{\mathcal{H}}| = \prod_{i=1}^{n} (s - \lambda_i) \left[\left(s + \lambda_j - \frac{\tilde{q}_{jj}}{s-\lambda_j} C_{jj}\right) \prod_{\substack{i=1 \\ i\neq j}}^{n} (s + \lambda_i)\right]$$

$$= \left[(s + \lambda_j)(s - \lambda_j) - \tilde{q}_{jj}C_{jj}\right] \prod_{\substack{i=1 \\ i\neq j}}^{n} (s + \lambda_i)(s - \lambda_i) = 0 \qquad (3\text{-}57)$$

where λ_i are the open loop eigenvalues. The eigenvalues of the matrix $\tilde{\mathcal{H}}$ are

$$\left.\begin{aligned} s_i &= \pm \lambda_i \quad i \neq j,\ i = 1, 2, \ldots n \\ s_j &= \pm \sqrt{\lambda_j^2 + \tilde{q}_{jj}\, C_{jj}} \end{aligned}\right\} \tag{3-58}$$

If the j^{th} closed-loop eigenvalue s_j is to be specified $\tilde{q}_{jj}$ can be determined from the above equation

$$\tilde{q}_{jj} = \frac{s_j^2 - \lambda_j^2}{c_{jj}} \tag{3-59}$$

The optimal control law is given by

$$u = -R^{-1}\, G^T\, T^{-T}\, \tilde{p} = -R^{-1}\, G^T\, T^{-T}\, \tilde{P}_{\mathcal{Z}} = \tilde{K}\, T^{-1}\, x \tag{3-60}$$

where $\tilde{K}$ is the optimal feedback gain in the transformed domain and the $\tilde{P}$ is the solution of the Riccati equation

$$\tilde{P}\Lambda + \Lambda\tilde{P} + \tilde{Q} - \tilde{P}\, T^{-1}\, GR^{-1}\, G^T\, T^{-T}\, \tilde{P} = 0 \tag{3-61}$$

In the optimal feedback system

$$\dot{x} = [F + GK]\, x \tag{3-62}$$

where

$$K = \tilde{K}T^{-1}$$

One pole of the open-loop system has been shifted to a specified location. Starting with the new system $F_1 = [F + GK]$, a performance index Q can be determined to shift the next pole. And in this sequential manner a weighting matrix Q can be determined to shift each pole and these different weighting matrices may be added together to get one weighting matrix which would shift all the eigenvalues to their specified locations. The procedure described here is applicable for systems with real distinct poles. If the system has complex poles, then, as stated in Reference 2, the system of equations can be transformed to the block diagonal form and some constraining equations have to be satisfied to shift the complex poles.

This sequential procedure for pole placement is extended to the determination of the movement of the zeros at each stage as the matrix Q is varied. Two sequential procedures, one computing the Riccati solution from a set of linear equations and the other computing the closed-loop eigenvectors, are presented in the following that determine the pole-zero movements as Q is varied for a given R matrix.

3.3.2 Design Procedure 1

This procedure is based upon obtaining the Riccati solution from a set of linear equations as described above. The procedure is applicable for systems with real distinct poles. For systems with complex poles, as stated above, some constraining equations have to be satisfied. To determine the movement of the zeros, the inverse of the matrix $[sI-\tilde{\mathcal{H}}]$ is required. From Equation (3-54a), the inverse can be written as

$$[sI-\tilde{\mathcal{H}}]^{-1} = \begin{bmatrix} sI-\Lambda & C \\ 0 & sI+\Lambda-\tilde{Q}[sI-\Lambda]^{-1}C \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -\tilde{Q}[sI-\Lambda]^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \qquad (3\text{-}63)$$

where the matrices A_i are given by

$$A_1 = [sI-\Lambda]^{-1} + [sI-\Lambda]^{-1} C\left\{sI+\Lambda-\tilde{Q}[sI-\Lambda]^{-1}C\right\}^{-1}\tilde{Q}[sI-\Lambda]^{-1}$$

$$A_2 = -[sI-\Lambda]^{-1} C\left\{sI+\Lambda-\tilde{Q}[sI-\Lambda]^{-1}C\right\}^{-1}$$

$$A_3 = -\left\{sI+\Lambda-\tilde{Q}[sI-\Lambda]^{-1}C\right\}^{-1}\tilde{Q}[sI-\Lambda]^{-1}$$

$$A_4 = \left\{sI+\Lambda-Q[sI-\Lambda]^{-1}C\right\}^{-1} \qquad (3\text{-}64)$$

The matrices A_i can be expressed in closed form in terms of the weighting matrix elements. To get the matrices A_i in closed form, it is required to get the inverse of the matrix $\left\{sI+\Lambda-\tilde{Q}[sI-\Lambda]^{-1}C\right\}$ in closed form. The inverse can be found from the technique given in Appendix A. With only the $\tilde{q}_{jj}$ element in $\tilde{Q}$ being non-zero.

$$\left\{sI+\Lambda-\tilde{Q}\left[sI-\Lambda\right]^{-1}C\right\}$$

$$= \begin{bmatrix} s+\lambda_1 & 0 & \cdots & & & 0 \\ 0 & s+\lambda_2 & \cdots & & & 0 \\ \vdots & & & & & \\ \frac{-\tilde{q}_{jj}C_{j1}}{s-\lambda_j} & \frac{-\tilde{q}_{jj}C_{j2}}{s-\lambda_j} & \cdots & s+\lambda_j\ \frac{-\tilde{q}_{jj}C_{jj}}{s-\lambda_j} & \cdots & \frac{-\tilde{q}_{jj}C_{jn}}{s-\lambda_j} \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & & s+\lambda_n \end{bmatrix}$$

and

$$\left\{sI+\Lambda-\tilde{Q}\ \ sI-\Lambda^{-1}\ C\right\}^{-1} =$$

$$\begin{bmatrix} \frac{1}{s+\lambda_1} & 0 & & 0 & \cdots & & 0 \\ 0 & \frac{1}{s+\lambda_2} & & 0 & \cdots & & \vdots \\ \vdots & & & & & & \vdots \\ \frac{\tilde{q}_{jj}C_{j1}}{(s+\lambda_1)a} & \frac{\tilde{q}_{jj}C_{j2}}{(s+\lambda_2)a} & \cdots & \frac{s-\lambda_j}{a} & \frac{\tilde{q}_{jj}C_{jj+1}}{(s+\lambda_{j+1})a} & \cdots & \frac{\tilde{q}_{jj}C_{jn}}{(s+\lambda_n)a} \\ & & & & \frac{1}{s+\lambda_{j+1}} & & \\ & & & & & \ddots & \\ & & & & & & \frac{1}{s+\lambda_n} \end{bmatrix} \tag{3-65}$$

where

$$a = (s+\lambda_j)(s-\lambda_j) - \tilde{q}_{jj}C_{jj} \tag{3-66}$$

The matrices A_1 and A_2 are needed to get closed-loop transfer functions and are now given by

$$A_1 = \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & \cdots & \frac{C_{1j}\tilde{q}_{jj}}{a(s-\lambda_1)} & \cdots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \cdots & \frac{C_{2j}\tilde{q}_{jj}}{a(s-\lambda_2)} & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & \cdots & \frac{C_{jj}\tilde{q}_{jj}}{a(s-\lambda_j)} + \frac{1}{s-\lambda_j} & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & \cdots & \frac{C_{nj}\tilde{q}_{jj}}{a(s-\lambda_n)} & \cdots & \frac{1}{s-\lambda_n} \end{bmatrix} \tag{3-67}$$

and A_2 can be written as a sum of two matrices

$$A_2 = C_1 + C_2 \tag{3-68}$$

where

$$C_1 = \begin{bmatrix} \frac{C_{11}}{(s-\lambda_1)(s+\lambda_1)} & \frac{C_{12}}{(s-\lambda_1)(s+\lambda_2)} & \cdots & \frac{C_{1j}(s-\lambda_j)}{a(s-\lambda_1)} & \cdots & \frac{C_{1n}}{(s-\lambda_1)(s+\lambda_n)} \\ \frac{C_{21}}{(s-\lambda_2)(s+\lambda_1)} & \cdots & \cdots & \frac{C_{1j}(s-\lambda_j)}{a(s-\lambda_2)} & \cdots & \frac{C_{2n}}{(s-\lambda_2)(s+\lambda_n)} \\ \vdots & & & & & \\ \frac{C_{n1}}{(s-\lambda_n)(s+\lambda_1)} & \cdots & \cdots & \frac{C_{nj}(s-\lambda_j)}{a(s-\lambda_n)} & \cdots & \frac{C_{nn}}{(s-\lambda_n)(s+\lambda_n)} \end{bmatrix} \tag{3-69}$$

$$
C_2 = \begin{bmatrix}
\frac{C_{1j}\tilde{q}_{jj}C_{j1}}{a(s-\lambda_1)(s+\lambda_1)} & \frac{C_{1j}\tilde{q}_{jj}C_{j2}}{a(s-\lambda_1)(s+\lambda_2)} & \cdots\,0 & \frac{C_{1j}\tilde{q}_{jj}C_{jj+1}}{a(s-\lambda_1)(s+\lambda_{j+1})} & \cdots & \frac{C_{1j}\tilde{q}_{jj}C_{jn}}{a(s-\lambda_1)(s+\lambda_n)} \\
\frac{C_{2j}\tilde{q}_{jj}C_{j1}}{a(s-\lambda_2)(s+\lambda_1)} & \frac{C_{2j}\tilde{q}_{jj}C_{j2}}{a(s-\lambda_2)(s+\lambda_2)} & \cdots\,0 & \frac{C_{2j}\tilde{q}_{jj}C_{jj+1}}{a(s-\lambda_2)(s+\lambda_{j+1})} & \cdots & \frac{C_{2j}\tilde{q}_{jj}C_{jn}}{a(s-\lambda_2)(s+\lambda_n)} \\
\vdots & & & & & \\
\frac{C_{nj}\tilde{q}_{jj}C_{j1}}{a(s-\lambda_n)(s+\lambda_1)} & \frac{C_{nj}\tilde{q}_{jj}C_{j2}}{a(s-\lambda_n)(s+\lambda_2)} & \cdots\,0 & \frac{C_{nj}\tilde{q}_{jj}C_{jj+1}}{a(s-\lambda_n)(s+\lambda_{j+1})} & \cdots & \frac{C_{nj}\tilde{q}_{jj}C_{jn}}{a(s-\lambda_n)(s+\lambda_n)}
\end{bmatrix}
\tag{3-70}
$$

That part of the adjoint matrix required to get the closed-loop transfer functions can be obtained by multiplying the matrices A_1 and A_2 by $\Delta(s)\,\Delta(-s)$. The submatrices B_1 and B_2 of the matrix adj $[sI-\mathcal{H}]$ are given by

$$
B_1 = \Delta(s)\,\Delta(-s)\,A_1
$$

$$
= \begin{bmatrix}
a\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s-\lambda_i) & \cdots & C_{1j}\tilde{q}_{jj}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s-\lambda_i) & \cdots & 0 \\
0 \quad a\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 2,j}}^{n}(s-\lambda_i) & - & C_{2j}\tilde{q}_{jj}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 2,j}}^{n}(s-\lambda_i) & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & (s+\lambda_j)\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s-\lambda_i) & \cdots & 0 \\
\vdots & & \vdots & & \\
0 & \cdots & C_{nj}\tilde{q}_{jj}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s-\lambda_i) & \cdots & a\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s+\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s-\lambda_i) \\
\vdots & & & & \\
0 & \cdots & & \cdots &
\end{bmatrix}
\tag{3-71}
$$

$$B_2 = \Delta(s)\ \Delta(-s)\ C_1 + \Delta(s)\,\Delta(-s)\ C_2$$

and

$$C_1\ \Delta(s)\,\Delta(-s)$$

$$= \begin{bmatrix}
C_{11}a \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s-\lambda_i)(s+\lambda_i) & C_{12}a \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq 2,j}}^{n} (s+\lambda_i) & \cdots & C_{1j}(s-\lambda_j) \cdot \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s+\lambda_i) & \cdots & C_{1n}a \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s-\lambda_i) \cdot \prod\limits_{\substack{i=1\\ i\neq j,n}}^{n} (s+\lambda_i) \\
\vdots & & & & & \\
\dfrac{C_{j1}a}{(s-\lambda_j)} \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s+\lambda_i) & \dfrac{C_{j2}a}{s-\lambda_j} \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq 2,j}}^{n} (s+\lambda_i) & \cdots & C_{jj} \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s-\lambda_i)(s+\lambda_i) & \cdots & C_{jn}a \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s-\lambda_i) \cdot \prod\limits_{\substack{i=1\\ i\neq j,n}}^{n} (s+\lambda_i) \\
\vdots & & & & & \\
C_{n1}a \prod\limits_{\substack{i=1\\ i\neq n,j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n} (s+\lambda_i) & C_{n2}a \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq 2,j}}^{n} (s+\lambda_i) & \cdots & C_{nj}(s-\lambda_j) \prod\limits_{\substack{i=1\\ i\neq n,j}}^{n} (s-\lambda_i) \prod\limits_{\substack{i=1\\ i\neq j}}^{n} (s+\lambda_i) & \cdots & C_{nn}a \prod\limits_{\substack{i=1\\ i\neq j,n}}^{n} (s-\lambda_i) \cdot (s+\lambda_i)
\end{bmatrix}$$

(3-72)

$$
= \frac{C_2\,\Delta(s)\,\Delta(-s)}{}\begin{bmatrix}
C_{1j}\tilde{q}_{jj}C_{j1}\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s-\lambda_i)(s+\lambda_i)----0 & C_{1j}\tilde{q}_{jj}C_{jj+1}\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s-\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j,j+1}}^{n}(s+\lambda_i)--- & C_{1j}\tilde{q}_{jj}C_{jn}\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s-\lambda_i)\cdot\prod\limits_{\substack{i=1\\ i\neq n,j}}^{n}(s+\lambda_i) \\
\vdots & & \\
\frac{C_{jj}\tilde{q}_{jj}C_{j1}}{(s-\lambda_j)}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s-\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s+\lambda_i)--0 & \frac{C_{jj}\tilde{q}_{jj}C_{jj+1}}{(s-\lambda_j)}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s-\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j,j+1}}^{n}(s+\lambda_i)--- & \frac{C_{jj}\tilde{q}_{jj}C_{jn}}{S-\lambda_j}\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(s-\lambda_i)\cdot\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s+\lambda_i) \\
\vdots & & \\
C_{nj}\tilde{q}_{jj}C_{j1}\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s-\lambda_i)\prod\limits_{\substack{i=1\\ i\neq 1,j}}^{n}(s+\lambda_i)--0 & C_{nj}\tilde{q}_{jj}C_{nj+1}\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s-\lambda_i)\prod\limits_{\substack{i=1\\ i\neq j,j+1}}^{n}(s+\lambda_i)--- & C_{nj}\tilde{q}_{jj}C_{jn}\prod\limits_{\substack{i=1\\ i\neq j,n}}^{n}(s-\lambda_i)\cdot(s+\lambda
\end{bmatrix}
$$

(3-73)

The matrices B_1 and B_2 are functions of the performance index matrices Q and R. The closed loop transfer functions are as given before:

$$
x(s) = \frac{T(B_1(s) - B_2(s)\,\tilde{P})\,Gu_c(s)}{\Delta(s)} \tag{3-74}
$$

where P can be computed as follows:

$$
\tilde{P} = B_2^{-1}\,B_1\Big|_{s=s_i} \qquad s_i = \lambda_i \qquad i \neq j
$$

and s_j is defined by Equation (3-66).

Equations (3-66) and the numerator polynomials of (3-74) determine the movement of the j^{th} pole and in general all the zeros as $\tilde{Q}$ is varied. The Q matrix can be obtained from $\tilde{Q}$ and starting with the new system $F_1 = [F + GK]$, where K is the feedback gain corresponding to Q, the procedure is repeated to determine the movement of other poles and zeros.

In this manner the movement of all the poles and zeros can be determined as performance index weighting matrices are varied. It is to be noted that if the system one starts with is an optimal system, where the feedback gain, K_o, is the result of minimizing a performance index using a weighting matrix Q_o and if some of the poles and zeros in this optimal system are not at the desired locations, the present design method may be used to shift them to more desirable locations and the performance index matrix Q and the gain matrix K are added to Q_o and K_o to get the total Q and K. This step by step design procedure for determining the movement of poles and individual transfer function zeros is summarized in the following steps:

1. The system of equations is transformed into Jordan form using the transformation matrix T.

2. The matrices B_1 and B_2 which are submatrices of the matrix adj $[sI-\tilde{\mathcal{H}}]$ are obtained. With only $\tilde{q}_{jj}$ being non-zero in the matrix $\tilde{Q}$, the Riccati solution is determined.

3. Equation (3-66) along with B_1 and B_2 determine the movement of one pole and, in general, all the zeros.

4. With $\tilde{Q}_1 = \{(\tilde{q}_{jj})\}$ the optimal feedback gain can be computed as $\tilde{K} = - R^{-1} G^T T^{-T}\tilde{P}$ in the transformed domain or after transforming back into the original coordinates the feedback gain can be computed as $K = - R^{-1} G^T T^{-T} \tilde{P}T^{-1}$.

5. The matrix Q in the original coordinates is obtained as $Q = T^{-T} Q_1 T^{-1}$.

6. Starting now with the new system $F_1 = (F-GK)$, steps 1 through 4 are repeated to determine the movements of the next pole and all the zeros. The final Q and K can be obtained by adding the weighting matrices and the feedback gains at different stages.

It is to be noted that after all the poles are shifted and the zero movements determined, if some pole-zero locations are not desirable, the present method can again be used to move around the poles and zeros and the Q and K matrices can be added to previous Q and K matrices to get the final result. An example will illustrate the ideas involved in the design procedure.

<u>Example</u>

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the performance index is given by

$$J = \frac{1}{2}\int_0^\infty (x^TQx + u^TRu)dt$$

1. The first step is to transform the system into Jordan or diagonal form using the transformation

$$x = T\mathfrak{z}$$

where

$$T = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

The eigenvalues of F are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.

The transformed equations are given by

$$\dot{\mathfrak{z}} = T^{-1} FT\mathfrak{z} + T^{-1} G$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$J = \frac{1}{2} \int_0^\infty (z^T \tilde{Q} z + u^T Ru)dt$$

The matrix $C = T^{-1} GR^{-1} G^T T^{-T}$, $\tilde{Q} = T^T QT$ and the matrix R was chosen to be

$$R = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix C is now given by

$$C = \begin{bmatrix} 8 & -6 & 5 \\ -6 & \frac{32}{7} & \frac{-27}{7} \\ 5 & \frac{-27}{7} & \frac{23}{7} \end{bmatrix}$$

2. Let the matrix $\tilde{Q}$ be

$$\tilde{Q} = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with this $\tilde{Q}$ matrix only the first pole will be shifted and the defining expression for shifting the first pole is given by

$$(\lambda_{c_1} - \lambda_1)(\lambda_{c_1} + \lambda_1) - c_{11}\tilde{q}_{11} = 0$$

or

$$(\lambda_{c_1} + 1)(\lambda_{c_1} - 1) - 8\tilde{q}_{11} = 0$$

3. The matrix B_1 is given by

$$B_1(s) = \begin{bmatrix} \tilde{q}_{11}(s-2)(s-3)(s+2)(s+3)(s-1) & 0 & 0 \\ -6\tilde{q}_{11}(s-2)(s-3)(s+3) & (s^2-1-8\tilde{q}_{11})(s-2)(s-3)(s+3) & \\ 5\tilde{q}_{11}(s-2)(s-3)(s+2) & 0 & (s^2-1-8\tilde{q}_{11})(s-2)(s-3)(s+2) \end{bmatrix}$$

The matrix B_2 is given by

$$B_2(s) = C_1\,\Delta(s)\,\Delta(-s) + C_2\,\Delta(s)\,\Delta(-s)$$

where

$$C_1\Delta(s)\Delta(-s) = \begin{bmatrix} 8(s+2)(s+3)(s-2)(s-3) & \frac{-6}{s+1}(s^2-1-8\tilde{q}_{11})(s+2)(s+3)(s-3) & \frac{5}{s+1}(s^2-1-8\tilde{q}_{11})(s+2)(s+3)(s-2) \\ -6(s+1)(s+3)(s-2)(s-3) & \frac{32}{7}(s^2-1-8\tilde{q}_{11})(s+3)(s-3) & \frac{-27}{7}(s^2-1-8\tilde{q}_{11})(s+3)(s-2) \\ 5(s+1)(s+2)(s-2)(s-3) & \frac{-27}{7}(s^2-1-8\tilde{q}_{11})(s+2)(s-3) & \frac{23}{7}(s^2-1-8\tilde{q}_{11})(s+2)(s-2) \end{bmatrix}$$

$$C_2\Delta(s)\Delta(-s) = \begin{bmatrix} 0 & \frac{-48\tilde{q}_{11}}{s+1}(s+2)(s+3)(s-3) & \frac{40\tilde{q}_{11}}{s+1}(s+2)(s+3)(s-2) \\ 0 & 36\tilde{q}_{11}(s+3)(s-3) & -30\tilde{q}_{11}(s+3)(s-2) \\ 0 & -30\tilde{q}_{11}(s+2)(s-3) & 25\tilde{q}_{11}(s+2)(s-2) \end{bmatrix}$$

The matrices B_1 and B_2 are expressed as a function of $\tilde{q}_{11}$ and the chosen R. The pole-zero movements can be determined from $\Delta(s)$, $B_1(s)$ and $B_2(s)$ as $\tilde{q}_{11}$ is varied.

Suppose it is required to move the first pole from -1 to -5. From the defining expression for shifting of the first pole

$$\tilde{q}_{11} = \frac{\lambda_{c_1}^{\;2} - \lambda_1^{\;2}}{8} = 3$$

Substitution of s = 5 in the first row, s = 2 in the second row and s = 3 in the third of the matrices yields

$$B_1\Big|_{s=s_i \text{ in the } i^{th} \text{ row}} \begin{bmatrix} 1344 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_2\Big|_{\substack{s=s_i \text{ in the} \\ i^{th} \text{ row}}} \begin{bmatrix} 2688 & -672(24-8\tilde{q}_{11}) - \frac{5376\tilde{q}_{11}}{6} & \frac{840}{5}(24-\tilde{q}_{11}) + \frac{6720\tilde{q}_{11}}{6} \\ 0 & \frac{-160}{7}(3-8\tilde{q}_{11}) - 900\tilde{q}_{11} & 0 \\ 0 & 0 & \frac{115}{7}(8-8\tilde{q}_{11}) + 125\tilde{q}_{11} \end{bmatrix}$$

The Riccati solution $\tilde{P}$ is given by

$$\tilde{P} = B_2^{-1} B_1\Big|_{\substack{s=s_i \text{ in the} \\ i^{th} \text{ row}}} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The closed-loop transfer functions are given by

$$x(s) = \frac{T\left[B_1(s) - B_2(s)\,\tilde{P}\right]}{\Delta(s)} T^{-1} G\, u_c(s)$$

where u_c is the command input and $\Delta(s)$ is the characteristic polynomial of the optimal system.

The performance index matrix Q corresponding to the above transfer functions is given by

$$Q = T^{-T}\,\tilde{Q}\,T^{-1}$$

where

$$\tilde{Q} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the optimal gain matrix K corresponding to Q is given by

$$K = \tilde{K}\, T^{-1}$$

where $\tilde{K}$ is obtained from $\tilde{P}$.

Thus far, a performance index matrix Q has been found to determine the movement of one pole and all the zeros.

The same procedure can be used to determine the movement of other poles and zeros starting with the new system matrix $F_1 = F - GK$. The performance index matrix Q and the gain matrix K can be added to the previous Q and K to get the total Q and K.

3.3.3 Design Procedure 2

The eigenvectors of the optimal closed-loop system can be used to determine the movement of the closed-loop zeros of the transfer functions. The development given here is for systems with real distinct eigenvalues. The system of equations (3-1) is transformed into Jordan form and the Hamiltonian set of equations for the transformed system is as given before:

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \Lambda & -C \\ -\tilde{Q} & -\Lambda \end{bmatrix} \begin{bmatrix} z \\ \tilde{p} \end{bmatrix} = \tilde{\mathcal{H}} \begin{bmatrix} z \\ \tilde{p} \end{bmatrix}$$

The optimal closed-loop system and the adjoint can again be transformed into Jordan form with the transformation

$$\begin{bmatrix} \mathfrak{z} \\ \tilde{p} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{bmatrix} = V \begin{bmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{bmatrix} \tag{3-75}$$

where V_{ij} are nxn matrices and the first n columns of the matrix V correspond to the eigenvectors of the left-half plane eigenvalues and the last n columns correspond to the eigenvectors of the right-half plane eigenvalues. The transformed equations are given by

$$\begin{bmatrix} \dot{\mathfrak{z}}_1 \\ \dot{\mathfrak{z}}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_c & 0 \\ 0 & -\Lambda_c \end{bmatrix} \begin{bmatrix} \mathfrak{z}_1 \\ \mathfrak{z}_2 \end{bmatrix} \tag{3-76}$$

where Λ_c is a diagonal matrix of closed-loop eigenvalues.

The transfer functions $\frac{\mathfrak{z}(s)}{u_c(s)}$ can be written as

$$\frac{\mathfrak{z}(s)}{u_c(s)} = V_{11} \frac{\text{adj} \left[sI - \Lambda_c \right]}{\Delta(s)} V_{11}^{-1} \quad T^{-1} G \tag{3-77}$$

and the transfer functions $\frac{x(s)}{u_c(s)}$ is given by

$$\frac{x(s)}{u_c(s)} = T\, V_{11} \frac{\text{adj} \left[sI - \Lambda_c \right]}{\Delta(s)} V_{11}^{-1} \quad T^{-1} G \tag{3-78}$$

where $\Delta(s)$ is closed-loop characteristic polynomial of the optimal system and T is the transformation matrix required to transform the system of equations (3-1) into Jordan form.

The eigenvectors can be computed from the Hamiltonian matrix as follows:

$$\begin{bmatrix} \Lambda & -C \\ & \\ -\tilde{Q} & -\Lambda \end{bmatrix} \nu = \lambda_c \nu \tag{3-79}$$

where λ_c is a closed-loop eigenvalue and $\nu = \left[\nu_{11}\ \nu_{21}\ \ldots\ V_{2n1}\right]^T$ is an eigen-vector. With only one element $\tilde{q}_{11}$ of the matrix Q being non-zero,the first eigen-vector ν can be obtained from the following set of equations

$$\begin{bmatrix} \Lambda & -C \\ & \\ -\tilde{Q} & -\Lambda \end{bmatrix} \begin{bmatrix} \nu_{11} \\ \vdots \\ \nu_{2n1} \end{bmatrix} = \lambda_{c_1} \begin{bmatrix} \nu_{11} \\ \vdots \\ \nu_{2n1} \end{bmatrix} \tag{3-80}$$

where the eigenvalue λ_1 is changed to λ_{c_1}.

The above equation can be written as

$$\begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 & -C_{11} & -C_{12} & \cdots & -C_{1n} \\
0 & \lambda_2 & 0 & \cdots & 0 & -C_{21} & -C_{22} & \cdots & -C_{2n} \\
\vdots & & & & & & & & \\
0 & 0 & \cdots & & \lambda_n & -C_{n1} & -C_{n2} & \cdots & -C_{nn} \\
-\tilde{q}_{11} & 0 & \cdots & & 0 & -\lambda_1 & 0 & \cdots & 0 \\
\vdots & & & & & & & & \\
0 & 0 & \cdots & & 0 & 0 & \cdots & \lambda_j \cdots & 0 \\
0 & \cdots & & & 0 & 0 & \cdots & & -\lambda_n
\end{bmatrix}
\begin{bmatrix} \nu_{11} \\ \nu_{21} \\ \\ \nu_{n1} \\ \nu_{n+11} \\ \\ \\ \nu_{2n1} \end{bmatrix}
= \lambda_{c_1}
\begin{bmatrix} \nu_{11} \\ \nu_{21} \\ \\ \nu_{n1} \\ \nu_{n+11} \\ \\ \\ \nu_{2n1} \end{bmatrix}$$

(3-81)

resulting in 2n linear equations in 2n unknowns of the elements of the eigenvector ν. Each equation in the set of equations (3-81) can be written individually as follows

$$\begin{aligned}
\lambda_1 \nu_{11} - C_{11} \nu_{n+11} - C_{12} \nu_{n+21} \cdots\cdots - C_{1n} \nu_{2n1} &= \lambda_{c_1} \nu_{11} \\
\lambda_2 \nu_{21} - C_{21} \nu_{n+11} - C_{22} \nu_{n+21} \cdots\cdots - C_{2n} \nu_{2n1} &= \lambda_{c_1} \nu_{21} \\
&\vdots \\
\lambda_n \nu_{n1} - C_{n1} \nu_{n+11} - C_{n2} \nu_{n+21} \cdots\cdots - C_{nn} \nu_{2n1} &= \lambda_{c_1} \nu_{n1} \\
- \tilde{q}_{11} \nu_{11} - \lambda_1 \nu_{n+11} &= \lambda_{c_1} \nu_{n+11} \\
- \lambda_2 \nu_{n+21} &= \lambda_{c_1} \nu_{n+21} \\
&\vdots \\
- \lambda_n \nu_{2n1} &= \lambda_{c_1} \nu_{2n1}
\end{aligned} \tag{3-82}$$

The last (n-1) equations can be written as

$$\begin{aligned}
(\lambda_{c_1} + \lambda_2) \nu_{n+21} &= 0 \\
(\lambda_{c_1} + \lambda_3) \nu_{n+31} &= 0 \\
&\vdots \\
(\lambda_{c_1} + \lambda_n) \nu_{2n1} &= 0
\end{aligned} \tag{3-83}$$

since $(\lambda_c + \lambda_i) \neq 0$ for i = 2, ... n, then to satisfy the set of equations (3-83) v_{j1}, j = n+2, ... 2n must be equal to zero. The element v_{n+11} can be expressed in terms of v_{11} as

$$v_{n+11} = \frac{-\tilde{q}_{11}}{(\lambda_1 + \lambda_{c_1})} v_{11} \tag{3-84}$$

Substituting for v_{j1}, j = n+1, ... 2n in the first n equations in the set of equations (3-82) results in the following set of n equations

$$\lambda_1 v_{11} + C_{11} \frac{\tilde{q}_{11}}{\lambda_1 + \lambda_c} v_{11} = \lambda_{c_1} v_{11}$$

$$\lambda_2 v_{21} + C_{21} \frac{\tilde{q}_{11}}{\lambda_1 + \lambda_c} v_{11} = \lambda_{c_1} v_{21}$$

$$\lambda_n v_n + C_{n1} \frac{\tilde{q}_{11}}{\lambda_1 + \lambda_c} v_{11} = \lambda_{c_1} v_{n1} \tag{3-85}$$

The first equation in the set of equations (3-85) can be written as

$$\left[(\lambda_{c_1} - \lambda_1)(\lambda_{c_1} + \lambda_1) - C_{11}\tilde{q}_{11}\right] v_{11} = 0$$

The quantity multiplying v_{11} in the above equation is identically equal to zero. This fact can be seen from equation (3-57). The expression

$$(s + \lambda_1)(s - \lambda_1) - C_{11}\tilde{q}_{11} = 0$$

defines the movement of the first pole, λ_{c_1}, as $\tilde{q}_{11}$ is varied. Substitution of λ_{c_1} for s yields the expression

$$(\lambda_{c_1} + \lambda_1)(\lambda_{c_1} - \lambda_1) - C_{11}\tilde{q}_{11} = 0 \tag{3-86}$$

The element v_{11} of the eigenvector can now be chosen arbitrarily. The elements v_{i1}, i = 2, ... n are given by

$$v_{i1} = \frac{C_i \tilde{q}_{11}}{(\lambda_{c_1} + \lambda_1)(\lambda_{c_1} - \lambda_i)} v_{11} \tag{3-87}$$

The closed-loop eigenvalues λ_{c_i}, i = 2, ... n are the same as the open-loop eigenvalues λ_i, i = 2, ... n. The eigenvector corresponding to the eigenvalue λ_{c_2} can be computed from the following set of equations

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & -C_{11} & -C_{12} & \cdots & -C_{1n} \\ 0 & \lambda_2 & 0 & \cdots & 0 & -C_{21} & -C_{22} & \cdots & -C_{2n} \\ \vdots & & & & & & & & \\ 0 & \cdots & & & \lambda_n & -C_{n1} & -C_{n2} & \cdots & -C_{nn} \\ -\tilde{q}_{11} & \cdots & & & 0 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 & 0 & -\lambda_2 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & \cdots & & & 0 & \cdots & & & -\lambda_n \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \\ v_{n+12} \\ \vdots \\ v_{2n2} \end{bmatrix} = \lambda_{c_2} \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \\ v_{n+12} \\ \vdots \\ v_{2n2} \end{bmatrix} \tag{3-88}$$

The set of equations (3-88) can be written individually as follows:

$$\lambda_1 \nu_{12} - C_{11} \nu_{n+12} - C_{12} \nu_{n+22} \cdots\cdots -C_{1n} \nu_{2n2} = \lambda_{c_2} \nu_{12}$$

$$\lambda_2 \nu_{22} - C_{21} \nu_{n+12} - C_{22} \nu_{n+22} \cdots\cdots -C_{2n} \nu_{2n2} = \lambda_{c_2} \nu_{22}$$

$$\vdots$$

$$\lambda_n \nu_{n2} - C_{n1} \nu_{n+12} - C_{n2} \nu_{n+22} \cdots\cdots -C_{nn} \nu_{2n2} = \lambda_{c_2} \nu_{n2}$$

$$\begin{aligned} - \tilde{q}_{11} \nu_{12} - \lambda_1 \nu_{n+12} &= \lambda_{c_2} \nu_{n+12} \\ - \lambda_2 \nu_{n+22} &= \lambda_{c_2} \nu_{n+22} \\ &\vdots \\ - \lambda_n \nu_{2n2} &= \lambda_{c_2} \nu_{2n2} \end{aligned} \tag{3-89}$$

The elements ν_{i2}, i = n+2, ... 2n should be equal to zero to satisfy the set of equations (3-89). The element ν_{n+12} is given by

$$\nu_{n+12} = \frac{-q_{11}}{\lambda_{c_2} + \lambda_1} \nu_{12}$$

Substitution of ν_{n+12} in the first equation of the set of equations (3-89) yields

$$\lambda_1 \nu_{12} + C_{11} \frac{\tilde{q}_{11}}{\lambda_{c_2} + \lambda_1} \nu_{12} = \lambda_{c_2} \nu_{12}$$

or

$$\left[(\lambda_{c_2} - \lambda_1)(\lambda_{c_2} + \lambda_1) - C_{11} \tilde{q}_{11}\right] \nu_{12} = 0 \tag{3-90}$$

Since λ_{c_2} is the same as the open-loop eigenvalue λ_2, the quantity multiplying v_{12} in equation (3-90) cannot be equal to zero. Therefore, to satisfy equation (3-90) v_{12} must equal zero. From the second equation of the set of equation (3-89) it follows that

$$\lambda_2 \, v_{22} = \lambda_{c_2} \, v_{22}$$

and hence v_{22} can be arbitrary.

The elements of the eigenvector corresponding to the eigenvalue λ_{c_2} are now given by

$$v_{i2} = 0, \; i = 1, \ldots, 2n, \; i \neq 2 \qquad (3\text{-}91)$$

In a very similar manner, the elements of the eigenvectors corresponding to the remaining eigenvalues can be obtained and are given by

$$\left.\begin{array}{l} v_{ij} = 0, \; i \neq j \\ v_{ij} = \text{arbitrary value } i=j \end{array}\right\} \begin{array}{l} i = 1, \ldots, 2n \\ j = 1, \ldots, 2n \end{array} \qquad (3\text{-}92)$$

Setting the elements of the eigenvector which are arbitrary to 1, the eigenvector matrix V_{11} is given by

$$V_{11} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_n & 0 & & & 1 \end{bmatrix} \qquad (3\text{-}93)$$

where

$$a_i = \frac{C_{i1} \, \tilde{q}_{11}}{(\lambda_{c_1} + \lambda_1)(\lambda_{c_1} - \lambda_i)} \quad i = 2, \ldots, n \qquad (3\text{-}94)$$

From Appendix B, the inverse of V_{11} can be written as

$$V_{11}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_2 & 1 & & \vdots \\ -a_3 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ -a_n & 0 & & 1 \end{bmatrix} \tag{3-95}$$

The closed-loop transfer functions, $\frac{z}{u_c}(s)$, are now given by

$$\frac{z(s)}{u_c(s)} = V_{11} \left[sI - \Lambda_c\right]^{-1} V_{11}^{-1} \, G_z$$

where $$G_z = T^{-1} G \text{ and } x = T z$$

$$\Lambda_c = \begin{bmatrix} \lambda_{c_1} & 0 & \cdots & 0 \\ 0 & \lambda_{c_2} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_{c_n} \end{bmatrix} \tag{3-96}$$

and the open-loop eigenvalue λ_1 is changed to λ_{c_1} and $\lambda_{c_i} = \lambda_i$, $i = 2, \ldots, n$.

The inverse of the matrix $\left[sI - \Lambda_c\right]$ can be written as

$$\left[sI - \Lambda_c\right]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} \prod\limits_{\substack{i=1 \\ i\neq 1}}^{n} (s-\lambda_{c_i}) & 0 & \cdots & 0 \\ 0 & \prod\limits_{\substack{i=1 \\ i\neq 2}}^{n} (s-\lambda_{c_i}) & \cdots & 0 \\ & & \ddots & \\ & & & \prod\limits_{\substack{i=1 \\ i\neq n}}^{n} (s-\lambda_{c_i}) \end{bmatrix} \tag{3-97}$$

where $\Delta(s)$ is the closed-loop characteristic polynomial of the optimal system.

Let the i^{th} diagonal element of the matrix $\left[sI - \Lambda_c\right]^{-1}$ be denoted by b_i. The transfer functions can now be written as

$$\frac{\zeta}{u_c(s)}(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_2 & 1 & & \\ \vdots & & & \\ a_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & & \\ \vdots & & & \\ 0 & & \cdots & b_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_2 & 1 & & 0 \\ \vdots & & & \vdots \\ -a_n & 0 & \cdots & 1 \end{bmatrix} G_\zeta$$

$$= \frac{1}{\Delta(s)} \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ a_2 b_1 - a_2 b_2 & b_2 & & \vdots \\ a_3 b_1 - a_3 b_3 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ a_n b_1 - a_n b_n & & & b_n \end{bmatrix} G_\zeta \qquad (3\text{-}98)$$

The closed-loop transfer functions of the x states are given by

$$X(s) = \frac{T}{\Delta(s)} \begin{bmatrix} b_1 & 0 & \cdots & & 0 \\ a_2(b_1 - b_2) & b_2 & 0 & \cdots & 0 \\ a_3(b_1 - b_3) & 0 & b_3 & & \\ \vdots & & & & \\ a_n(b_1 - b_n) & 0 & \cdots & & b_n \end{bmatrix} T^{-1} G\, u_c(s) \qquad (3\text{-}99)$$

The a_i are explicit functions of $\tilde{q}_{11}$ and hence the coefficients of numerator polynomials are an explicit function of $\tilde{q}_{11}$.

Equation (3-86) along with the polynomials b_i and the eigenvector elements a_i determine the movement of the first pole and all the zeros. The performance index matrix Q and the corresponding gain matrix K can be obtained from

$$
\begin{aligned}
Q &= T^{-T}\,\tilde{Q}\,T^{-1} \\
K &= \tilde{K}\,T^{-1}
\end{aligned}
\tag{3-100}
$$

where $\tilde{K}$ is gain matrix corresponding to $\tilde{Q}$.

Starting now with the new system $F_1 = F - GK$, where K is the gain matrix associated with the performance index matrix Q, the movement of other poles and zeros can be determined. The procedure is summarized in the following steps:

1. The system of equations is transformed into Jordan or diagonal form using the transformation matrix T_1.

2. With only the $\tilde{q}_{11}$ element of the matrix $\tilde{Q}_1$ being non-zero, the matrix V_{11} is determined.

3. Equation (3-86) along with the matrix V_{11} determine the movement of the first pole and, in general, all the zeros as q_{11} is varied. The coefficients of the numerator polynomials are an explicit function of $\tilde{q}_{11}$.

4. The weighting matrix Q_1 and the corresponding gain matrix K_1 are determined from $Q_1 = T_1^{-T}\,\tilde{Q}_1\,T_1^{-1}$ and $K_1 = \tilde{K}_1\,T_1^{-1}$, respectively.

5. Starting with the new system matrix, $F_1 = F - GK_1$, steps 1 through 4 are repeated to determine the movements of other poles and zeros and a new performance index matrix Q_2 and gain matrix K_2 are obtained and added to the previous performance index and gain matrices Q_1 and K_1 to get the total

$Q = Q_1 + Q_2$ and total $K = K_1 + K_2$ to move the open-loop poles and zeros to the closed-loop locations determined by Q and K. This procedure is repeated until all the poles and zeros are moved and the performance index and the gain matrices at each stage is added to the previous total to get total Q and K at that stage.

It is to be noted that if, after all the poles and zeros are moved, some of pole-zero locations are not desirable, this procedure can again be used to move the poles and zeros to move desirable locations and the Q and K can be added to previous Q and K to get the new Q and K.

In summary, the pole-zero movements are determined through the closed-loop eigenvectors as the performance index matrix is varied. It is of interest to note that in Ref. 6, performance index matrices are constructed based on model design specifications. But the problem addressed there is the characterization of performance index matrices in terms of asymptotic eigenvalues and eigenvectors.

A simple example will illustrate the ideas involved. The example is the same one as given before.

Example

The F and G matrices are as given before

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1. With the transformation matrix

$$T = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

The system is transformed into the Jordan or diagonal form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

2. The weighting matrix on the control and states are

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix $C = T^{-1}\, GR^{-1}\, G^T\, T^{-T}$ is given by

$$C = \begin{bmatrix} \frac{49}{r_1} + \frac{1}{r_2} & \frac{-35}{r_1} - \frac{1}{r_2} & \frac{28}{r_1} + \frac{1}{r_2} \\ \frac{-35}{r_1} - \frac{1}{r_2} & \frac{25}{r_1} + \frac{1}{r_2} & \frac{-20}{r_1} - \frac{1}{r_2} \\ \frac{28}{r_1} + \frac{1}{r_2} & \frac{-20}{r_1} - \frac{1}{r_2} & \frac{16}{r_1} + \frac{1}{r_2} \end{bmatrix}$$

The defining expression for shifting the first pole is given by

$$(\lambda_{c_1} - \lambda_1)(\lambda_{c_1} + \lambda_1) - C_{11}\, \tilde{q}_{11} = 0$$

or

$$\lambda_{c_1}^2 - 1 - \left(\frac{49}{r_1} + \frac{1}{r_2}\right) \tilde{q}_{11} = 0$$

3. The eigenvector matrix V_{11} is given by

$$V_{11} = \begin{bmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{bmatrix}$$

$$a_2 = \frac{\left(\frac{-35}{r_1} - \frac{1}{r_2}\right)\tilde{q}_{11}}{(\lambda_{c_1} - 1)(\lambda_{c_1} + 2)}$$

$$a_3 = \frac{\left(\frac{28}{r_1} + \frac{1}{r_2}\right)\tilde{q}_{11}}{(\lambda_{c_1} - 1)(\lambda_{c_1} + 3)}$$

$$V_{11}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ -a_3 & 0 & 1 \end{bmatrix}$$

4. The closed-loop transfer functions are given by

$$x(s) = T\, V_{11} \left[sI - \Lambda_c\right]^{-1} V_{11}^{-1}\, T^{-1}\, G\, u_c(s)$$

where

$$\Lambda_c = \begin{bmatrix} -\lambda_{c_1} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\left[sI - \Lambda_c\right]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} (s+2)(s+3) & 0 & 0 \\ 0 & (s-\lambda_{c_1})(s+3) & 0 \\ 0 & 0 & (s-\lambda_{c_1})(s+2) \end{bmatrix}$$

where $\Delta(s)$ is the characteristic polynomial of the optimal system.

$$x(s) = \frac{T}{\Delta(s)} \begin{bmatrix} s^2 + 5s + 6 & 0 & 0 \\ \dfrac{\left(\frac{-35}{r_1} - \frac{1}{r_2}\right)\tilde{q}_{11} \left\{s^2+5s+6-(s-\lambda_{c_1})(s+3)\right\}}{(\lambda_{c_1}-1)(\lambda_{c_1}+2)} & (s-\lambda_{c_1})(s+3) & 0 \\ \dfrac{\left(\frac{28}{r_1} + \frac{1}{r_2}\right)\tilde{q}_{11} \left\{s^2+5s+6-(s-\lambda_{c_1})(s+2)\right\}}{(\lambda_{c_1}-1)(\lambda_{c_1}+3)} & 0 & (s-\lambda_{c_1})(s+2) \end{bmatrix} T^{-1}G$$

Carrying out the indicated multiplications, the x transfer functions can be obtained as explicit functions of $\tilde{q}_{11}$, r_1 and r_2. The pole-zero movements can be determined as $\tilde{q}_{11}$ is varied for given r_1 and r_2.

If it is desired to move the first pole from -1 to -5, the $\tilde{Q}$ and R matrices are given by

$$\tilde{Q} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvector matrix is given by

$$V_{11} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{5}{4} & 0 & 1 \end{bmatrix}$$

$$V_{11}\,[sI - \Lambda_c]^{-1}\, V_{11}^{-1}\, T^{-1}\, G$$

$$= \begin{bmatrix} 7(s^2+5s+6) & 0 & 0 \\ -5(s^2+8s+15) + 7(3s+9) & s^2+8s+15 & 0 \\ 4(s^2+7s+10) - 7(\frac{5}{2}s+5) & 0 & s^2+7s+10 \end{bmatrix} T^{-1}G$$

The x transfer functions can be obtained by multiplying the above matrix by T.

5. The weighting matrix Q can be computed from $\tilde{Q}$ as

$$Q = T^{-T}\tilde{Q}T^{-1} = \begin{bmatrix} \frac{243}{4} & \frac{135}{4} & \frac{27}{2} \\ \frac{135}{4} & \frac{75}{4} & \frac{15}{2} \\ \frac{27}{2} & \frac{15}{2} & 3 \end{bmatrix}$$

6. The new system matrix F_1 is computed as

$$F_1 = F - GK$$

where K corresponds to the weighting matrix Q and the new system has poles at

$$\lambda_{c_1} = -5, \quad \lambda_{c_2} = -2, \quad \lambda_{c_3} = -3$$

and the numerator polynomials are as given before.

Starting with the new system the same procedure is used to move the other poles and zeros.

Section 4
ALTERNATIVE DESIGN TECHNIQUES

In this section two alternative design techniques are presented for pole-zero placement. In the first design technique to be presented, the pole-zero movements are determined as a function of the Riccati solution P and the weighting matrix R on the control rather than as a function of Q, the weighting matrix on the states. In the second design technique to be presented, the differential behavior of the zeros and poles are determined under perturbations in Q and R. This analysis linearizes the dependence of the coefficients of the numerator and denominator polynomials of the transfer functions on the perturbations in the weighting matrices Q and R. This analysis could be used as a basis for iterative and computational techniques to study the behavior of zeros and poles as a function of Q and R.

4.1 OPTIMAL CONTROL DESIGN IN TERMS OF P AND R

The concern here is to generate state feedback matrices that result from a performance index without solving the algebraic Riccati equation involved. In other words, given the familiar optimal control problem

$$\dot{x} = Fx + Gu, \; y = Hx$$

$$J = \frac{1}{2} \int_0^\infty (x^T Qx + u^T Ru)dt$$

it is required to generate all the possible state feedback matrices K defining the optimal control. It is hoped to be able to find such feedback matrices without having to solve the algebraic Riccati equation

$$PF + F^T P - PGR^{-1} G^T P + Q = 0$$

for P, where P is needed to find the gain matrix

$$K = - R^{-1} G^T P.$$

It is shown here that, for stable F, if P is positive semidefinite, then so is Q.

Some of the facts used to establish the positive semidefiniteness of Q given positive semidefinite P and positive definite R are given in the following remarks:

1. If Q is a sum of two matrices Q_1 and Q_2, where Q_1 and Q_2 are positive definite, then Q is also positive definite. If both Q_1 and Q_2 are positive semidefinite, then so is Q.

2. If L is a real matrix, then $L^T L$ is positive semidefinite.

3. Given any positive definite matrix Q_1 and stable F, there exists a symmetric, positive definite matrix P, if and only if, it is the unique solution of the set of $\frac{n(n+1)}{2}$ linear equations

$$F^T P + PF = -Q_1 \tag{4-1}$$

4. From remark 3 it follows that if F has eigenvalues in the left half plane, then for any positive definite P there exists a unique positive definite matrix Q_1.

5. From the continuity of the eigenvalues of a matrix as a function of its entries, if P of equation (4-1) is positive semidefinite, then so is Q_1.

6. The matrix $PGR^{-1}G^TP$ can be expressed as MM^T where $M = PGR^{-1/2}$ and $R^{-1/2}$ is the inverse of the positive definite square root of R. By letting $MM^T = Q_2$, where from remark 2 Q_2 is positive semidefinite, the Riccati equation can now be written using equation (4-1) as

$$Q_1 + Q_2 = Q \tag{4-2}$$

From remarks 1 through 6, it follows that given positive definite matrices P and R and the F matrix with eigenvalues in the left half plane, the matrix Q in the Riccati equation is positive definite. If P is positive semidefinite, so is Q.

The closed-loop transfer function matrix becomes

$$W(s) = H \left\{ sI - \left[F - GR^{-1} G^T P \right] \right\}^{-1} G \tag{4-3}$$

where

$$y(s) = W(s)\ u_c(s)$$

and u_c is the command input.

The closed-loop transfer functions are expressed in terms of R and P and it is guaranteed that the positive semidefinite P could result only from a positive semidefinite Q and hence the feedback gain matrix K could result only from a performance index and hence optimal.

4.2 DIFFERENTIAL BEHAVIOR OF ZEROS AND POLES

The first order changes in the individual transfer function zeros and system poles under perturbations in the performance index matrices Q and R are determined. The perturbations in the state feedback matrix K under perturbations in Q and R are also determined. Firstly, the effect of perturbing Q on K is considered.

<u>Perturbation in Q</u>

The perturbation, dP, in the Riccati solution P is determined when Q is perturbed by dQ. From the Riccati equation, it follows that

$$dP \left[F - GR^{-1} G^T P \right] + \left[F^T - PGR^{-1} G^T \right] dP = -\, dQ \tag{4-4}$$

Also from the equation defining the gain matrix K the perturbation dK in K is given by

$$dK = -R^{-1} G^T dP \tag{4-5}$$

From equation (4-4) it follows that

$$dP \left[F + GK \right] + \left[F^T + K^T G^T \right] dP = - dQ \tag{4-6}$$

or

$$dP F_k + F_k^T dP = - dQ \tag{4-7}$$

where

$$F_k = F + GK \tag{4-8}$$

and then dK is found from equation (4-5).

Perturbation in R

The first order approximation of $\left[R + dR \right]^{-1}$ is given by

$$\left[R + dR \right]^{-1} = R^{-1} - R^{-1} dR R^{-1} \tag{4-9}$$

From equation (4-9) and the equation defining K it follows that

$$dPF_k + F_k^T dP = - K^T dR K \tag{4-10}$$

Total Perturbation of K

The total perturbation of K as a function of dQ and dR is then determined by

$$dP F_k + F_k^T dP = - dQ - K^T dR K \tag{4-11}$$

which is a linear expression in dP, and then dK is obtained as

$$dK = - R^{-1} \left[G^T dP + dR\ K \right] \tag{4-12}$$

This is an explicit expression of dK as a linear combination of dQ and dR. Equations (4-11) and (4-12) could be used as the basis of iterative computational algorithm to both solve the Riccati equation and find the feedback matrix K. In addition, it gives an insight into the behavior of K as a function of Q and R.

Effect of Perturbation on Poles and Zeros

Let

$$W(s) = H \left[sI - F_k \right]^{-1} G \tag{4-13}$$

be the transfer function matrix of the system under state feedback where

$$F_k = F + GK$$

The perturbation transfer function matrix

$$W(s) + dW(s) = H \left[sI - (F_k + GdK) \right]^{-1} G \tag{4-14}$$

$$= \frac{H\ \mathrm{adj} \left[sI - (F_k + GdK) \right]}{\left| \left[sI - (F_k + GdK) \right] \right|}$$

Since only the first order perturbations are of interest, it is clear from equation (14) that the coefficients of all the polynomials in W(s) are perturbed linearly by the different entries of dK.

A study of the differential behavior of K as a function of Q and R has been presented. This study leads to linear explicit expressions of dK and also shows the behavior of individual transfer function zeros and system poles as a result of perturbation in Q and R.

Section 5
CONTROL SYSTEMS DESIGN

In this section a control systems design is presented using the X-22A V/STOL airplane as the model. The first design procedure presented in Section 3 is used for the control systems design.

5.1 MATHEMATICAL MODEL

The small perturbation longitudinal linearized equations of motion can be represented in a general form as

$$\dot{x} = Fx + Gu \quad (1)$$

with

$$x = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} \qquad u = \begin{bmatrix} \delta_{e_s} \\ \delta_t \end{bmatrix}$$

$\Delta u = u - u_o$, $\Delta w = w - w_o$, $\Delta q = q - q_o$ and $\Delta\theta = \theta - \theta_o$ where u_o, w_o, q_o and θ_o are reference values and

- u - velocity along body x-axis (ft/sec)
- w - velocity along body z-axis (ft/sec)
- q - body axis pitch rate (deg/sec, rad/sec)
- θ - pitch attitude (deg, rad)
- δ_{e_s} - input which is designed to produce a pitching moment
- δ_t - input which produces thrust

The F and G matrices at an airspeed of 65 knots are given below:

$$F = \begin{bmatrix} -.18 & -.03 & 9.57 & -31.87 \\ -.2 & -.55 & 109.43 & 2.78 \\ -.01 & -.0177 & -.09 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} -.356 & .52 \\ 0 & -1 \\ .33 & .021 \\ 0 & 0 \end{bmatrix}$$

The open-loop transfer functions with respect to both the inputs are given in Table 5-1. The longitudinal dynamics of the X-22A at an airspeed of 65 knots are typical of unaugmented VTOL aircraft in this flight regime. The short period dynamics are characterized by lightly damped complex roots while the phugoid, which is normally oscillatory and stable in high speed flight has degenerated into a real pair, one of which is unstable. Because of this unstable root, the unaugmented X-22A would not meet the level requirement of the specification MIL-F-83300 for longitudinal dynamic response.

An augmentation concept which has produced satisfactory pilot ratings under instrument flight conditions in the X-22A is attitude command augmentation. This augmentation system employs pitch rate and attitude feedback to the pitching moment controller to augment the frequency and damping of the short period roots and to stabilize the aperiodic roots. To achieve the desired objectives the first design procedure described in Section 3 was used.

The first step is to transform the equations of motion of the X-22A into diagonal form. With the transformation

TABLE 5-1

OPEN-LOOP TRANSFER FUNCTION CHARACTERISTICS

CHARACTERISTIC EQUATION

$$s^4 + .82s^3 + 2.19s^2 + .0735s - .0544$$

Poles

Real	Imaginary	ζ	w
-.389	1.426	.263	1.48
-.389	-1.426		
.138			
-.1806			

ZEROS OF TRANSFER FUNCTIONS

Transfer Functions	Zeros Real	Zeros Imaginary
$\frac{u}{\delta_{es}}$	- .48	
	4.36	3.86
	4.36	-3.86
$\frac{w}{\delta_{es}}$	- .099	.231
	- .099	- .231
$\frac{q}{\delta_{es}}$	0	
	- .576	
	- .165	
$\frac{\theta}{\delta_{es}}$	- .576	
	- .165	
$\frac{u}{\delta_t}$	.723	
	- .9	1.26
	- .9	-1.26
$\frac{w}{\delta_t}$	.006	.48
	.006	- .48
	1.91	

TABLE 5-1 (CONT'D)

Transfer Functions	Zeros Real	Zeros Imaginary
$\frac{q}{\delta_t}$	0	
	-1.17	
	-1.56	
$\frac{\theta}{\delta_t}$	-1.17	
	-1.56	

$$x = T_1 z$$

where

$$x = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} \begin{bmatrix} 22.758 & -12.709 & -195.55 & 273.48 \\ 32.998 & -95.562 & 107.05 & -154.64 \\ 1.3363 & .28934 & .26646 & -.02582 \\ -.04901 & -.9241 & 1.93 & .143 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

the system of equation (1) is transformed into the block diagonal form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -.3887 & 1.426 & 0 & 0 \\ -1.426 & -.3887 & 0 & 0 \\ 0 & 0 & .138 & 0 \\ 0 & 0 & 0 & -.181 \end{bmatrix} + G \begin{bmatrix} \delta_{e_s} \\ \delta_t \end{bmatrix} \tag{2}$$

where

$$G_z = T_1^{-1} G = \begin{bmatrix} .216 & .012 \\ .097 & .012 \\ .05 & .006 \\ .021 & .0055 \end{bmatrix}$$

The performance index to be minimized is

$$J = \frac{1}{2} \int_0^\infty (z^T \tilde{Q}_1 z + u^T Ru)dt$$

In the first step the two complex poles, corresponding to the short period mode, are moved to more desirable locations. The short period natural frequency and the damping ratio were specified to be

$$\omega_{s_p} = 2 \pm .5 \text{ rad/sec}$$
$$\zeta_{s_p} = .6 \pm .2$$

The weighting matrix on the control was chosen for convenience to be

$$R = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

A different choice of R would result in a different Q matrix to achieve the design objectives.

The matrix $\tilde{Q}_1$ was selected so that the closed-loop short period natural frequency and the damping ratio would be in the desired range. The matrix $\tilde{Q}_1$ was determined to be

$$\tilde{Q}_1 = \begin{bmatrix} 25.173 & 0 & 0 & 0 \\ 0 & 25.173 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the short period characteristics are given by

$$\omega_{s_p} = 1.59$$
$$\zeta_{s_p} = .443$$

The matrix Q_1 of the performance index

$$J = \frac{1}{2} \int_0^\infty (x^T Q_1 x + u^T Ru)dt$$

is given by

$$Q_1 = T^{-T} \tilde{Q}_1 T^{-1}$$

The gain matrix K_1 corresponding to Q_1, the matrix Q_1, the closed-loop matrix $F_1 = F - GK_1$ and the transfer function characteristics are given in Table 5-2.

Thus far, the two complex poles have been moved to more desirable locations and the zero movement determined. It is to be noted from Table 5-2 that the open-loop unstable pole has been shifted to its mirror image in the left half plane. This is the property of optimal control design. The stable open-loop pole has remained at the same location. The zeros of the transfer

TABLE 5-2

F_1, Q_1 and K_1 MATRICES AND THE CORRESPONDING TRANSFER FUNCTIONS

$$F_1 = \begin{bmatrix} -.184 & -.037 & 10.51 & -30.93 \\ -.201 & -.551 & 109.5 & 2.88 \\ -.006 & -.011 & -.99 & -.93 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -.0131 & -.021 & 2.72 & 2.81 \\ -5.2 \times 10^{-4} & -8.43 \times 10^{-4} & .061 & .092 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} .0006 & .001 & .009 & .003 \\ .001 & .00197 & -.0198 & .0058 \\ -.0091 & -.0198 & 12.97 & -.162 \\ .003 & .0058 & -1.62 & .21 \end{bmatrix}$$

TRANSFER FUNCTION CHARACTERISTICS

Characteristic Equation

$$s^4 + 1.72s^3 + 2.98s^2 + .836s + .063$$

Poles

Real	Imaginary	ζ	w
-.702	1.42	.443	1.59
-.702	-1.42		
-.138			
-.181			

Zeros of Transfer Functions

Transfer Function	Zeros Real	Imaginary
$\frac{u}{\delta_{es}}$	-.484	
	4.34	3.88
	4.34	-3.88
$\frac{w}{\delta_{es}}$	-.0996	.231
	-.0996	-.231

TABLE 5-2 (CONT'D)

Transfer Function	Zeros Real	Zeros Imaginary
$\frac{q}{\delta_{es}}$	0	
	- .576	
	- .165	
$\frac{\theta}{\delta_{es}}$	- .576	
	- .165	
$\frac{u}{\delta_t}$	.295	
	-1.16	1.04
	-1.16	1.04
$\frac{w}{\delta_t}$	.0357	
	.494	.936
	.494	- .936
$\frac{q}{\delta_t}$	0	
	- .943	
	- .164	
$\frac{\theta}{\delta_t}$	- .943	
	- .164	

function of the pitch attitude to pitch control commands are affected by feedback from the second input, the thrust control command. It is seen from Table 5-2 that the elements of the second row of the feedback gain matrix corresponding to the feedback from the second input are small and have not moved the zeros of θ/δ_{e_s} significantly from their open-loop values.

In the next step one of the real poles, $\lambda = -.13805$, is moved close to one of the zeros of θ/δ_{e_s} transfer function. To do this, the new system of equations

$$\dot{x} = F_1 x + Gu$$

is transformed into block diagonal form again with the transformation

$$x = T_2 \mathfrak{z}$$

where

$$T_2 = \begin{bmatrix} -20.92 & 7.89 & 237.28 & 1018.7 \\ -55.55 & 71.22 & -108.9 & -576.05 \\ -.88 & -.824 & .297 & -.0962 \\ -.22 & .73 & -.215 & .532 \end{bmatrix}$$

The transformed equations are given by

$$\begin{bmatrix} \dot{\mathfrak{z}}_1 \\ \dot{\mathfrak{z}}_2 \\ \dot{\mathfrak{z}}_3 \\ \dot{\mathfrak{z}}_4 \end{bmatrix} = \begin{bmatrix} -.702 & 1.421 & 0 & 0 \\ -1.421 & -.702 & 0 & 0 \\ 0 & 0 & -.13805 & 0 \\ 0 & 0 & 0 & -.1806 \end{bmatrix} + G_{\mathfrak{z}} \begin{bmatrix} \delta_{e_s} \\ \delta_t \end{bmatrix}$$

where

$$G_{\mathfrak{z}} = \begin{bmatrix} -.24 & -.012 \\ -.155 & -.014 \\ -.184 & -.02 \\ .039 & .0051 \end{bmatrix}$$

The weighting matrix R is the same as chosen before. The matrix $\tilde{Q}_2$ was selected as

$$\tilde{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18.51 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the pole at -.13805 was moved to -.5762. The weighting matrix Q_2 corresponding to the x states is given by

$$Q_2 = T_2^{-T} \tilde{Q}_2 T_2^{-1}$$

The final closed-loop dynamic matrix $F_c = (F_1 - GK_2)$ where K_2 is the gain matrix corresponding to the weighting matrix Q_2, the final weighting matrix $Q = Q_1 + Q_2$, the final gain matrix K and the transfer function characteristics are given in Table 5-3.

It is to be noted that with sufficiently high feedback gains, one of the real poles and one of the zeros of Θ/δ_t transfer function are driven into close proximity to one of the zeros of the Θ/δ_{e_s} transfer function so that the aperiodic pair is close to the zeros of Θ/δ_{e_s} and Θ/δ_t transfer functions and the attitude response to pitch control commands is essentially second order.

The advantages of an attitude command augmentation system, compared to a simpler concept such as rate command augmentation, are

TABLE 5-3

F_c, Q and K MATRICES AND THE CORRESPONDING TRANSFER FUNCTIONS

$$F_c = \begin{bmatrix} -.191 & -.048 & 10.95 & -29.28 \\ -.201 & -.552 & 109.53 & 3.03 \\ .0011 & -1.57 \times 10^{-4} & -1.42 & -2.52 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -.034 & -.053 & 4.02 & 7.63 \\ -.0012 & -.0019 & .102 & .246 \end{bmatrix}$$

$$Q = \begin{bmatrix} .002 & .0033 & -.096 & -.321 \\ .0033 & .0053 & -.16 & -.5 \\ -.096 & -.16 & 18.5 & 18.9 \\ .321 & .5 & 18.9 & 76.41 \end{bmatrix}$$

TRANSFER FUNCTION CHARACTERISTICS

Characteristic Polynomial

$$s^4 + 2.16s^3 + 3.68s^2 + 2.05s + .261$$

Poles

Real	Imaginary	ζ	w
-.702	1.42	.442	1.59
-.702	-1.42		
-.181			
-.576			

Zeros of Transfer Functions

Transfer Function	Zeros Real	Zeros Imaginary
$\frac{u}{\delta_{es}}$	-.486	
	4.33	3.89
	4.33	-3.89
$\frac{w}{\delta_{es}}$	-.1	.232
	-.1	-.232

TABLE 5-3 (CONT'D)

Transfer Function	Zeros: Real	Zeros: Imaginary
$\frac{q}{\delta_{es}}$	0	
	-.577	
	-.165	
$\frac{\theta}{\delta_{es}}$	-.577	
	-.165	
$\frac{u}{\delta_t}$	-.606	.612
	-.606	- .612
$\frac{w}{\delta_t}$	-.246	
	.416	1.56
	.416	-1.56
$\frac{q}{\delta_t}$	0	
	-.58	
	-.2	
$\frac{\theta}{\delta_t}$	-.58	
	-.2	

(1) The pilot can estimate the magnitude of commanded attitude changes by the magnitude of his applied stick force or displacement.

(2) If unattended (stick force relaxed) the vehicle will return to the trim attitude.

(3) Rate feedback alone cannot stabilize an unstable aperiodic root.

The pole-zero locations of the Θ/δ_{e_s} and Θ/δ_t transfer functions at different stages in the design process are shown in Figure 5-1. Figure 5-1(a) shows the open-loop pole-zero locations. Figure 5-1(b) shows the pole-zero locations after the pair of complex poles corresponding to the short period roots are moved to more desirable locations. The use of the optimal control design procedure also moves the unstable pole to its mirror image in the left-half plane. The numerator zeros of Θ/δ_{e_s} have moved very little from their open-loop locations with the initial feedback. Figure 5-1(c) shows the final pole-zero locations and the two real poles approximately cancel the zeros of the Θ/δ_{e_s} and Θ/δ_t transfer functions. Thus, the attitude response to pitch control commands is essentially second order dominated by the modified short period mode. The responses of pitch attitude and pitch rate to pitch control step command are shown in Figure 5-2. The time history illustrates the second order nature of attitude response.

The design example presented in this section has demonstrated the usefulness of the step by step approach formulated in Section 3 to practical control systems design. The second design procedure discussed in Section 3 (based on the eigenvector approach) is considered to be more promising because of increased tractability of the modified system by the control systems designer. Additional study is required before it can be used in a systematic way for practical control systems design.

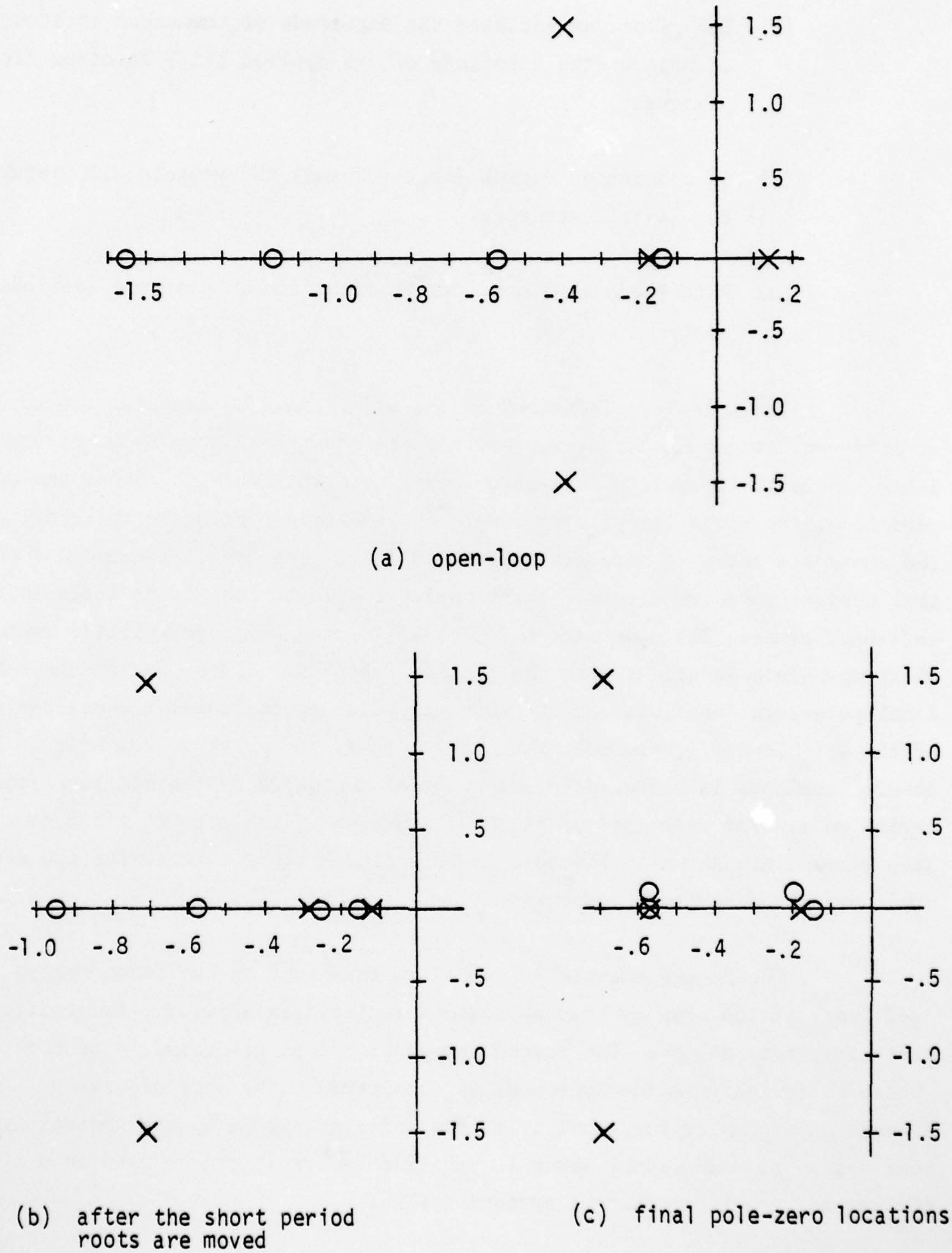


Fig. 5-1 POLE-ZERO LOCATIONS OF θ/δ_{es} AND θ/δ_t TRANSFER FUNCTIONS

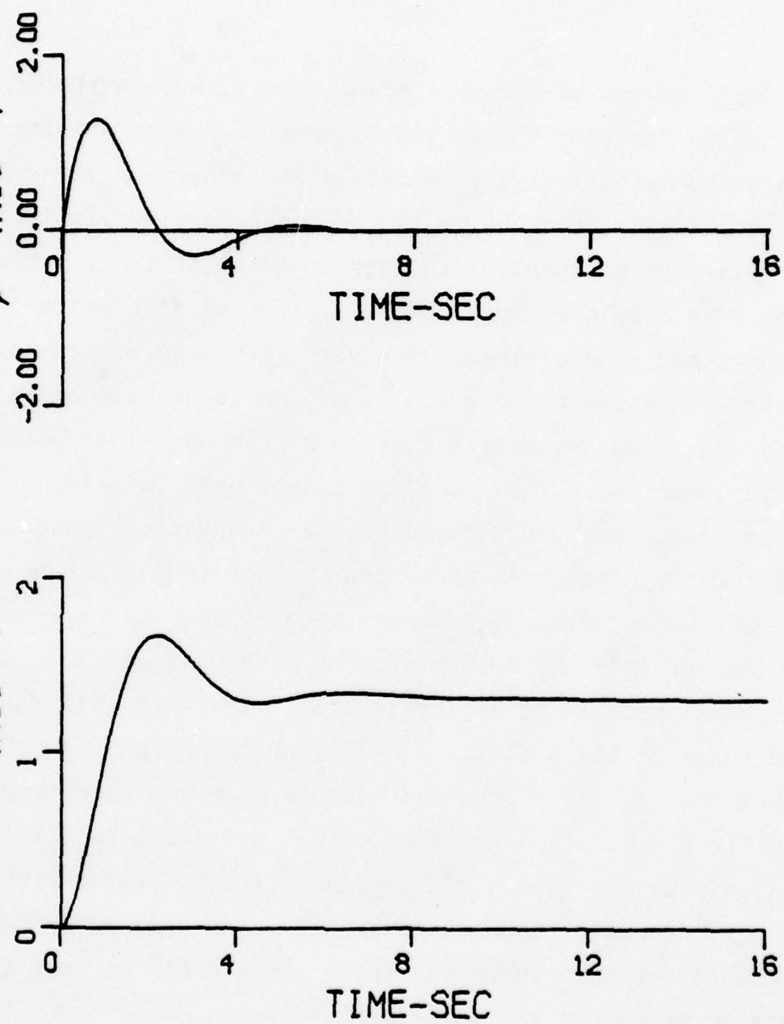


Fig. 5-2 PITCH RATE AND PITCH ATTITUDE RESPONSE

Section 6
SUMMARY AND RECOMMENDATIONS

Application of linear optimal control has developed several control system design procedures that can be used to design a multicontroller feedback system to satisfy flying qualities and other control systems design criteria. Specifically, two design procedures were given in Section 3 to determine the pole-zero movement as the performance matrix is varied. The procedures given were step-by-step procedures where at each stage a pole was moved and the zero movements determined. The performance index weighting matrices constructed at each stage to move the poles and zeros were added to get a final performance index weighting matrix that moves the open-loop poles and zeros to more desirable locations. The procedure can be used in as many steps as are required. The design procedure based on the eigenvector approach is the more promising of the two procedures presented in Section 3 since the closed-loop eigenvectors are computed at each stage to determine the zero movements and give better insight into the design process at each stage. In Section 4, two alternative design techniques were presented. The first technique was based on design in terms of the Riccati solution and the weighting matrix on the control rather than on the states and control. It was shown that design in terms of the Riccati solution and the control weighting matrix guaranteed that the feedback gain matrix result was optimal. In the second technique, the first order changes in transfer function poles and zeros were determined under perturbations in the performance index matrices. This could be used to develop an iterative design procedure for pole-zero placement.

In Section 5 the first design procedure developed in Section 3 was applied to a control system design using the X-22A V/STOL airplane as the model. The design was carried out in two stages to achieve attitude command augmentation. The example illustrated the ideas involved in the design procedure.

This study has shown the usefulness of the design procedures, based on the step-by-step approach, to practical control systems design. The pole-zero movements are determined directly at each step as the performance index matrix elements are varied. Further work is required to develop these design procedures into a viable control systems design tool, and to examine problems in optimal control systems design which have not been addressed in this study. The following recommendations for further work are therefore in order:

- The design procedure based on the eigenvector approach is considered to be more promising because of increased tractability of the modified system at every stage of the design process. Further work is required to develop the eigenvector approach into a more systematic procedure by using it to design practical control systems.

- The design techniques given in Section 4 need further development for use in control systems design.

- The effect of proximity of the transfer function zeros to the transmission zeros needs investiagtion. Since the transmission zeros are invariant under feedback, high feedback gains may be required to move a transfer function zero away from a transmission zero if required.

- Optimal control design technique requires full state feedback. In many practical situations, not all the states are available for feedback. Further work is required to determine the conditions under which optimal control design technique can be used with partial state feedback.

- An important practical problem in control systems design is the failure of a feedback path due to a sensor failure. The effect of this failure on the stability and transfer function

characteristics of the closed-loop multicontroller system needs to be investigated to examine the possibility of designing optimal control systems that remain stable under a wide class of failure modes.

REFERENCES

1. Whitbeck, R. F.: "A Comment on the Linear Optimal Control Problem", IEEE Trans. Automatic Control, Vol. AC-12, No. 6, pp. 777-778, December 1967.

2. Solheim, O. A.: "Design of Optimal Control Systems with Prescribed Eigenvalues", Int. J. Control, Vol. 15, No. 1, 143-160, 1972.

3. Rynaski, E. G and Whitbeck, R.F.: "The Theory and Application of Linear Optimal Control" Calspan Report No. 1H-1943-F-1, October 1965.

4. McRuer, D.T., Ashkenas, I.L. and Pass, H.R.: "Analysis of Multiloop Vehicular Control System" Air Force Flight Dynamics Laboratory Report No. ASD-TDR-62-1014, March 1964.

5. Ball, J.N. and Rynaski, E.G.: Longitudinal Flight Control for Military Aircraft - A Study of Requirements and Design Concepts" Calspan Report No. ID-1757-F-1, October 1963.

6. Harvey, C. A., Stein, G., and Doyle, J. G.: "Optimal Linear Control Characterization of Multi-Input Systems," Technical Report ONR CR 215-238-2, August 1977.

7. O'Donnell, J. J.: "Asymptotic Solution of the Matrix Riccati Equation of Optimal Control," Proc. Fourth Allerton Conference on Circuit and Systems Theory, pp. 577-586, University of Illinois, Urbana, Ill., October 5-7, 1966.

APPENDIX A

INVERSION OF A MATRIX OF A PARTICULAR FORM

It is required to invert a matrix A of the following form:

$$A = \begin{bmatrix} b_1 & 0 & 0 & - - - - - - - - - - - - & 0 \\ 0 & b_2 & 0 & - - - - - - - - - - - - & 0 \\ 0 & 0 & b_3 & - - - - - - - - - - - - & 0 \\ \vdots & & & & \\ a_1 & a_2 & a_3 & - - - \quad a_j + b_j - - - - & a_n \\ \vdots & & & & \\ 0 & 0 & & - - - - - - - - - - - - & b_n \end{bmatrix} \tag{1}$$

The inverse of the matrix can be found as the solution of a set of linear equations as

$$Ax = y \tag{2a}$$

$$x = A^{-1} y \tag{2b}$$

Equation (2a) can be written as

$$\begin{bmatrix} b_1 & 0 & 0 & - - - - - - - - - - & 0 \\ 0 & b_2 & 0 & - - - - - - - - - - & 0 \\ \vdots & & & & \\ a_1 & a_2 & a_3 & - - - \quad a_j + b_j \quad - - - & a_n \\ \vdots & & & & \\ 0 & 0 & & - - - - - - - - & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \tag{3}$$

The solution for x_i from the above set of equations is given by

$$x_1 = \frac{1}{b_1} y_1$$

$$x_2 = \frac{1}{b_2} y_2$$

$$\vdots$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots (a_j + b_j)\, x_j + \cdots\cdot a_n x_n = y_j$$

$$x_{j+1} = \frac{1}{b_{j+1}} y_{j+1} \tag{4}$$

$$\vdots$$

$$x_n = \frac{1}{b_n} y_n$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{1}{b_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{b_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{b_3} & \cdots & 0 \\ & & & & \\ \frac{-a_1}{b_1(a_j+b_j)} & \frac{-a_2}{b_2(a_j+b_j)} & \cdots & \frac{1}{(a_j+b_j)} \cdots & \frac{-a_n}{b_n(a_j+b_j)} \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & \frac{1}{b_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_j \\ \vdots \\ y_n \end{bmatrix}$$

$$= A^{-1} y \tag{5}$$

APPENDIX B

INVERSION OF A MATRIX OF A PARTICULAR FORM

The inversion of the following matrix is required:

$$A = \begin{bmatrix} 1 & 0 & - & - & - & 0 \\ a_2 & 1 & - & - & - & 0 \\ a_3 & 0 & 1 & - & - & 0 \\ | & & & & & | \\ | & & & & & | \\ | & & & & & | \\ a_n & 0 & - & - & - & 1 \end{bmatrix} \qquad (1)$$

The inverse can be obtained as the solution of the set of linear equations

$$\begin{bmatrix} 1 & 0 & - & - & - & 0 \\ a_2 & 1 & & & & | \\ | & & & & & | \\ | & & & & & | \\ | & & & & & | \\ a_n & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ | \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ | \\ | \\ | \\ y_n \end{bmatrix} \qquad (2)$$

The solution is given by

$$x_1 = y_1$$

$$a_2 x_1 + x_2 = y_2$$

or

$$x_2 = -a_2 y_1 + y_2$$

$$a_3 x_1 + x_3 = y_3$$

or

$$x_3 = -a_3 y_1 + y_3$$

The inverse of A can be written as

$$A^{-1} = \begin{bmatrix} 1 & 0 & - & - & - & - & - & - & - & 0 \\ -a_2 & 1 & - & - & - & - & - & - & & 0 \\ -a_3 & 0 & 1 & & - & - & - & & & 0 \\ | & & & & & & & & & \\ | & & & & & & & & & \\ | & & & & & & & & & \\ -a_n & 0 & - & - & - & - & - & - & - & 1 \end{bmatrix} \quad (3)$$

Distribution List

	Copies		Copies
Office of Naval Research 800 N. Quincy Street Arlington, VA 22217 R. von Husen, Code 211 S. L. Brodsky, Code 432	 4 1	David Taylor Naval Ship R&D Center Bethesda, MD 20084 W. E. Smith, Code 1576	 1
Office of Naval Research Branch Office 495 Summer Street Boston, MA 02210	 1	Naval Air Test Center Strike Aircraft Test Directorate Patuxent River, MD 20670 W. M. Branch	 1
Office of Naval Research Branch Office 715 Broadway - 5th Floor New York, NY 10003	 1	Naval Post Graduate School Monterey, CA 93940 Technical Reports Library	 1
Office of Naval Research Branch Office 536 South Clark St. Chicago, IL 60605	 1	Defense Documentation Center Building 5 Cameron Station Alexandria, VA 22314	 12
Office of Naval Research Branch Office 1030 E. Green St. Pasadena, CA 91106	 1	Air Force Office of Scientific Research Building 410 Bolling Air Force Base Washington, DC 20332 C. L. Nefzger	 1
Naval Research Laboratory Washington, DC 20375 Code 2627	 3	Air Force Flight Dynamics Laboratory Wright-Patterson Air Force Base Dayton, OH 45433 R. Anderson, Control Dyn. Br. F. George, Control Dyn. Br.	 1 1
Naval Air Systems Command Washington, DC 20361 G. Tsaparas, AIR 340D R. C. A'Harrah, AIR 53011	 1 1	Army Armament R&D Command Building #18 Dover, NJ 07801 N. Coleman, DRDAR-SCFCC	 1
Naval Air Development Center Warminster, PA 18974 C. R. Abrams, Code 6072 Technical Reports Library	 1 1	NASA Langley Research Center Hampton, VA 23665 Technical Library	 1
Naval Material Command Washington, DC 20360 Code 08T23	 1	Scientific Systems, Inc. Suite No. 309-310 186 Alewife Brook Parkway Cambridge, MA 02138 R. K. Mehra	 1
Naval Weapons Center China Lake, CA 93555 B. Hardy, Code 3914	 1		

	Copies
Systems Technology, Inc. 13766 South Hawthorne Blvd Hawthorne, CA 90250 R. Whitbeck	1
The Analytic Sciences Corp. 6 Jacob Way Reading, MA 01867 C. Price	1
Massachusetts Institute of Technology Lab for Information and Decision Systems Cambridge, MA 02139 M. Athans	1
University of Michigan Dept. of Naval Architecture & Marine Engr. Ann Arbor, MI 48109 M. G. Parsons	1
Nielsen Engineering & Research, Inc. 510 Clyde Ave. Mountainview, CA 94043 J. N. Nielsen	1
University of Notre Dame Dept. of Electrical Engineering Notre Dame, IN 46556 M. K. Sain	1
The C. S. Draper Laboratory, Inc. 555 Technology Square Cambridge, MA 02139 R. V. Ramnath	1
Honeywell, Inc. Systems and Research Center 2600 Ridgway Parkway Minneapolis, MN 55413 C. A. Harvey	1